MODERN INTRODUCTORY ANALYSIS

SCRATCHPAD 2

3. MATHEMATICAL INDUCTION
- SEQUENCES & SERIES

4. THE ALGEBRA OF VECTORS
NUMBER PAIRS AND GEOMETRY
ALGEBRA OF NUMBER PAIRS
...

3-2 SEQUENCES AND SERIES

In each case the first term of a sequence and a recursion formula for succeeding terms are given. If $n = \{1,2,3,4,5,6\}$, write the sequence.

1. $a_1 = 2$, $a_n + 1 = a_n - 1$: $2, 1, 0, -1, -2, -3$
2. $a_1 = 5$, $a_{n+1} = a_n - 3$: $5, 2, -1, -4, -7, -10$
3. $a_1 = -1$, $a_{n+1} = 3a_n$: $-1, -3, -9, -27, -81, -243$
4. $a_1 = 1$, $a_{n+1} = -2a_n$: $1, -2, 4, -8, 16, -32$
5. $a_1 = \frac{1}{2}$, $a_{n+1} = (-1)^{n+1} \cdot a_n$: $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
6. $a_1 = \frac{1}{3}$, $a_{n+1} = (-1)^n \cdot a_n$: $\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}$

7. $a_1 = 3$, $a_{n+1} = (a_n - 2)^2$: $3, 1, 1, 1, 1, 1$
8. $a_1 = 1$, $a_{n+1} = \frac{n}{n+1} \cdot a_n$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$

Write the first four terms of the sequence specified by the indicated rule. In each case, the initial value of the index is 1.

9. $a_k = 2k + 3$: $5, 7, 9, 11$

10. $a_k = \frac{2}{k}$: $2, 1, \frac{2}{3}, \frac{1}{2}$

11. $a_n = 4 \cdot n^2$: $4, 16, 36, 64$

(12) $a_n = \frac{n(n-1)}{2}$: 0, 1, 3, 6

(13) $a_t = |1-2^t|$: 1, 3, 7, 15

(14) $a_t = 3^{t+1}$: 9, 27, 81, 243

Specify each sequence (a) recursively
(b) by expressing a_n in terms of n.

(15) 2, 3, 4, 5 (a) $a_1 = 2$, $a_{n+1} = a_n + 1$

(b) $a_n = n+1$

(16) 2, 4, 8, 16 (a) $a_1 = 2$, $a_{n+1} = 2 \cdot a_n$

(b) $a_n = 2^n$

(17) 5, 10, 15, 20 (a) $a_1 = 5$, $a_{n+1} = a_n + 5$

(b) $a_n = 5 \cdot n$

(18) $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}$ (a) $a_1 = \frac{5}{2}$, $a_{n+1} = \frac{a_n}{2}$

(b) $a_n = \frac{5}{2^n}$

(19) $1, \frac{1}{1 \cdot 2}, \frac{1}{1 \cdot 2 \cdot 3}, \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$ (a) $a_1 = 1$, $a_{n+1} = \frac{a_n}{n+1}$

(b) $a_n = \frac{1}{n!}$

(20) $1, -1, -3, -5$ (a) $a_1 = 1$, $a_{n+1} = a_n - 2$;

(b) $a_n = n - 3(n-1) = n - 3n + 3 = 3 - 2n$

Write each series in expanded form.

(21) $\sum_{i=2}^{5} i = 2+3+4+5$

(22) $\sum_{n=3}^{6} |3-n| = 0+1+2+3$

(23) $\sum_{j=3}^{7} j = 3+4+5+6+7$

(24) $\sum_{k=2}^{5} (2k+1) = 5+7+9+11$

(25) $\sum_{t=1}^{5} (-1)^t \cdot t^2 = -1+4-9+16-25$

(26) $\sum_{i=1}^{3} (-1)^i (i^2 - i) = (-1)^1(1^2-1) + (-1)^2(2^2-2) + (-1)^3(3^2-3)$

$= 0 + 2 - 6$

(27) $\sum_{i=1}^{n} P(i) = P(1) + P(2) + \ldots + P(n)$

(28) $\sum_{j=0}^{n-1} P(j) = P(0) + P(1) + \ldots + P(n-1)$

of n.

$\frac{a_n}{n+1}$

Use the summation sign to write each series.

(29) $6+10+14+18 = \sum_{i=1}^{4} 6+4(i-1) = \sum_{i=1}^{4} 2+4i$

(30) $6-2+\frac{2}{3}-\frac{2}{9}$: THOUGHT PROCESSES SNAPSHOTS

Pattern $\frac{2}{3^{-1}}, \frac{2}{3^{0}}, \frac{2}{3^{1}}, \frac{2}{3^{2}}$

$-$ when n is even, $+$ when n is odd : $(-1)^{n+1}$ or $(-1)^{n-1}$

$$6-2+\frac{2}{3}-\frac{2}{9} = \sum_{i=1}^{4} (-1)^{i-1} \cdot \frac{2}{3^{i-2}}$$

or, better still, recognize this as a geometric series with first term 6, ratio $\left(-\frac{1}{3}\right)$

$$6-2+\frac{2}{3}-\frac{2}{9} = \sum_{i=1}^{4} 6\left(-\frac{1}{3}\right)^{i-1}$$

(31) $a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_i b_i$

(32) $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = \sum_{i=1}^{5} x_i^2$

I KNOW. SOME OF THIS IS "BABY MATH".

Find the sum of the series

(33) $\sum_{n=1}^{4} n^2 = 1+2^2+3^2+4^2 = 1+4+9+16 = 30$

(34) $\sum_{k=1}^{5} \frac{k(k+1)}{2} = 1+3+6+10+15 = 35$

(35) $\sum_{i=2}^{6} (8-i) = 6+5+4+3+2$

(36) $\sum_{t=-6}^{-4} \frac{t}{|t|} = \frac{-6}{|-6|} + \frac{-5}{|-5|} + \frac{-4}{|-4|} = -1-1-1 = -3$

Show that the following expressions name the same series.

(37) $\sum_{k=3}^{6} \left(\frac{k}{k+2}\right)$ and $\sum_{k=0}^{3} \left(\frac{k+3}{k+5}\right)$

$$\sum_{k=3}^{6} \left(\frac{k}{k+2}\right) = \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} = \sum_{k=0}^{3} \left(\frac{k+3}{k+5}\right)$$

(38) $\sum_{t=5}^{8} \left(\frac{t}{t+4}\right) = \frac{5}{9} + \frac{6}{10} + \frac{7}{11} + \frac{8}{12} = \sum_{t=1}^{4} \left(\frac{t+4}{t+8}\right)$

(39) $\sum_{k=6}^{9} \left(\frac{k}{k-1}\right) = \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} = \sum_{k=8}^{11} \left(\frac{k-2}{k-3}\right)$

(40) $\sum_{k=0}^{x} a_{k+1}$, $\sum_{k=1}^{x+1} a_k$, and $\sum_{k=t}^{x+t} a_{k-t+1}$

$$\sum_{k=0}^{x} a_{k+1} = a_1 + a_2 + \ldots + a_x + a_{x+1} = \sum_{k=1}^{x+1} a_k = a_1 + a_2 + a_3 + \ldots + a_{x+1}$$

$$\sum_{k=t}^{x+t} a_{k-t+1} = a_1 + a_{(t+1)-t+1} + a_{(t+2)-t+1} + \dots$$

$$+ a_{(x+t)-t+1} = a_1 + a_2 + a_3 + \dots + a_{x+1}$$

(41) What is the sum of the series $\sum_{k=1}^{n} (-1)^k$ if n is odd? What about if n is even?

Let's see. If n is odd: $\sum_{k=1}^{n} (-1)^k = -1+1-1+\dots-1$

The sum is -1.

If n is even, the sum is 0.

(42) If a, b, c are successive terms in a sequence, can you conclude that either $a < b < c$ or $a > b > c$? Justify your answer.

No. Neither can be concluded. See exercises 4, 5, 6.

(43) Write the nth term of the series $\sum_{k=3}^{100} (k^2 - 6k)$.

I will expand to see the first few terms to see if I can detect a pattern so as to derive a formula a_k in terms of k.

$$\sum_{k=3}^{n} (k^2 - 6k) = (9-18) + (16-24) + (25-30) + (36-36) + \dots + (n^2 - 6n)$$
$$= -9 - 8 - 5 - 0 + 7 + \dots + (n^2 - 6n)$$

The key to solving this problem is to note that the series starts at $k=3$, so the first term at $n=1$ has $k=3$; the second term at $n=2$ has $k=4$. How to deal with this in a mathematically precise manner?

?

$$\sum_{k=3}^{100} (k^2 - 6k) = \sum_{j=1}^{98} (j+2)^2 - 6(j+2), \text{ where } k = j+2.$$

For $j = n$: $(n+2)^2 - 6(n+2) = n^2 + 4n + 4 - 6n - 12$
$= n^2 - 2n - 8$

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44 If $\sum_{b=2}^{4} (a^2 b - ab) = \sum_{c=3}^{5} (ac+6)$, determine a

scratchwork:

$$\sum_{b=2}^{4} (a^2 b - ab) = (2a^2 - 2a) + (3a^2 - 3a) + (4a^2 - 4a)$$
$$= 9a^2 - 9a = 9a(a-1)$$

$$\sum_{c=3}^{5} (ac+6) = (3a+6) + (4a+6) + (5a+6)$$
$$= 12a + 18 = 6(2a+3)$$

$$9a^2 - 9a = 12a + 18$$
$$9a^2 - 21a - 18 = 0$$

FACTOR $ax^2 + bx + c$
$u \cdot v = ac = (9)(-18)$
$= -162$
$u + v = b = -21$

$\boxed{a = 9 \quad b = -21 \quad c = -18}$

Factors of -162: $(1)(162)$, $(2)(81)$, $(3)(54)$, $\boxed{(6)(27)}$

$-27 + 6 = -21$ so $u = -27$, $v = 6$ and $(-27)(6) = -162$

$ax^2 + ux + vx + c \rightarrow 9x^2 - 27x + 6x - 18 = 9x(x-3) + 6(x-3)$

FACTORS: $(9x+6)(x-3)$ $\therefore$ $9a^2 - 21a - 18 = 0$
$(9a+6)(a-3) = 0$

So, either $(9a+6)=0$ or $a-3=0$

$9a=-6$

$a=-\frac{6}{9}=-\frac{2}{3}$ or $a=3$

An easier way would have been to factor out 3 from $9a^2-21a-18=0$

$3(3a^2-7a-6)=0$

Then, for $3x^2-7x-6$, $a=3$, $b=-7$, $c=-6$

$u \cdot v = a \cdot c = (3)(-6) = -18$

$u+v=b=-7$

Factors of 18 : $(1)(18)$, $\boxed{(2)(9)}$, $(3)(6)$

$2-9=-7$ so $u=2$, $v=-9$, $u \cdot v=(2)(-9)=-18$

$3x^2-9x+2x-6 = 3x(x-3)+2(x-3)$

$=(3x+2)(x-3)$

$3a^2-7a-6=0$ when $3a+2=0$

$(3a+2)(a-3)=0$

$3a=-2$

$a=-\frac{2}{3}$

OR $a-3=0$

$a=3$ $\therefore a=-\frac{2}{3}, 3$

{45–50} By mathematical induction, prove that each assertion is true for every positive integer n.

4

and

(45) $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(1) $1 \in S : \sum_{k=1}^{1} 1^2 = 1 = \frac{1(1+1)(2 \cdot 1+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6}$

(2) Assume $x \in S$ and $\sum_{k=1}^{x} k^2 = \frac{x(x+1)(2x+1)}{6}$ is true

Then $$\sum_{k=1}^{x+1} k^2 = \frac{(x+1)(x+2)(2(x+1)+1)}{6} = \frac{(x+1)(x+2)(2x+3)}{6}$$

and $x+1 \in S$: $$\left(\sum_{k=1}^{x} k^2\right) + (x+1)^2 = \sum_{k=1}^{x+1} k^2 = \frac{x(x+1)(2x+1)}{6} + (x+1)^2$$

$$= \frac{x(x+1)(2x+1) + 6(x^2+2x+1)}{6}$$

$$= \frac{(x^2+x)(2x+1) + 6x^2 + 12x + 6}{6}$$

$$= \frac{2x^3 + x^2 + 2x^2 + x + 6x^2 + 12x + 6}{6} = \frac{2x^3 + 9x^2 + 13x + 6}{6}$$

$$= \frac{(x+1)(x+2)(2x+3)}{6}$$

(46) $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

(1) $1 \in S$: $\sum_{k=1}^{1} k^3 = 1^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2 = 1^2 = 1$

(2) Assume $x \in S$ and $\sum_{k=1}^{x} k^3 = \left[\frac{x(x+1)}{2}\right]^2$

Then $\sum_{k=1}^{x+1} k^3 = \left[\frac{(x+1)(x+2)}{2}\right]^2$ and $x+1 \in S$:

$$\sum_{k=1}^{x} k^3 + (x+1)^3 = \left[\frac{x(x+1)}{2}\right]^2 + (x+1)^3$$

$$= (x+1)^2 \cdot \left(\frac{x}{2}\right)^2 + (x+1)^3$$

$$= (x+1)^2\left[\left(\frac{x}{2}\right)^2 + (x+1)\right]$$

$$= (x+1)^2\left[\frac{x^2}{4} + x + 1\right] = (x+1)^2\left[\frac{x^2+4x+4}{4}\right]$$

$$= (x+1)^2\left[\frac{(x+2)^2}{2^2}\right] = \left[\frac{(x+1)(x+2)}{2}\right]^2$$

(47) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(1) $1 \in S$: $\sum_{i=1}^{1} (a_i + b_i) = a_1 + b_1 = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i = a_1 + b_1$

(2) Assume $x \in S$ and $\sum_{i=1}^{x} (a_i + b_i) = \sum_{i=1}^{x} a_i + \sum_{i=1}^{x} b_i$

Then $\sum_{i=1}^{x+1} (a_i + b_i) = \sum_{i=1}^{x+1} a_i + \sum_{i=1}^{x+1} b_i$ and $x+1 \in S$:

$$\sum_{i=1}^{x+1} (a_i + b_i) = \left[\sum_{i=1}^{x} (a_i + b_i) \right] + (a_{x+1} + b_{x+1})$$

$$= \left(\sum_{i=1}^{x} a_i + \sum_{i=1}^{x} b_i \right) + (a_{x+1} + b_{x+1})$$

$$= \left(\sum_{i=1}^{x} a_i + a_{x+1} \right) + \left(\sum_{i=1}^{x} b_i + b_{x+1} \right)$$

$$= \sum_{i=1}^{x+1} a_i + \sum_{i=1}^{x+1} b_i$$

(48) $\sum_{i=1}^{n} c \cdot a_i = c \sum_{i=1}^{n} a_i$

(1) $1 \in S$: $\sum_{i=1}^{1} c \cdot a_i = c \cdot a_1 = c \sum_{i=1}^{1} a_i = c \cdot a_1$

(2) Assume $x \in S$ and $\sum_{i=1}^{x} c \cdot a_i = c \sum_{i=1}^{x} a_i$

Then $\sum_{i=1}^{x+1} c \cdot a_i = c \sum_{i=1}^{x+1} a_i$ and $x+1 \in S$:

$$\sum_{i=1}^{x+1} c \cdot a_i = \sum_{i=1}^{x} c \cdot a_i + c \cdot a_{x+1}$$

$$= c \sum_{i=1}^{x} a_i + c \cdot a_{x+1}$$

$$= c\left[\left(\sum_{i=1}^{x} a_i\right) + a_{x+1}\right] = c \cdot \sum_{i=1}^{x+1} a_i$$

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(49) $\sum_{k=1}^{n} a = a \cdot n$

(1) $1 \in S$: $\sum_{k=1}^{1} a = a = 1 \cdot a$

(2) Assume $x \in S$ and $\sum_{k=1}^{x} a = a \cdot x$

Then $\sum_{k=1}^{x+1} a = a(x+1)$ and $x+1 \in S$:

$$\sum_{k=1}^{x+1} a = \sum_{k=1}^{x} a + a = ax + a = a(x+1)$$

(50) $$\sum_{j=1}^{n}(a_j+b_j)^2 = \sum_{j=1}^{n} a_j^2 + 2\sum_{j=1}^{n} a_j b_j + \sum_{j=1}^{n} b_j^2$$

(1) $1 \in S$: $$\sum_{j=1}^{1}(a_j+b_j)^2 = (a_1+b_1)^2 = a_1^2 + 2a_1b_1 + b_1^2 = \sum_{j=1}^{1} a_j^2 + 2\sum_{j=1}^{1} a_j b_j + \sum_{j=1}^{1} b_j^2$$

(2) Assume $x \in S$ and $$\sum_{j=1}^{x}(a_j+b_j)^2 = \sum_{j=1}^{x} a_j^2 + 2\sum_{j=1}^{x} a_j b_j + \sum_{j=1}^{x} b_j^2$$

Then $$\sum_{j=1}^{x+1}(a_j+b_j)^2 = \sum_{j=1}^{x+1} a_j^2 + 2\sum_{j=1}^{x+1} a_j b_j + \sum_{j=1}^{x+1} b_j^2$$

and $x \in 1 \in S$: $$\sum_{j=1}^{x+1}(a_j+b_j)^2$$

$$= \sum_{j=1}^{x}(a_j+b_j)^2 + (a_{x+1}+b_{x+1})^2$$

$$= \sum_{j=1}^{x} a_j^2 + 2\sum_{j=1}^{x} a_j b_j + \sum_{j=1}^{x} b_j^2 + a_{x+1}^2 + 2a_{x+1}b_{x+1}$$

$$+ b_{x+1}^2 = \left(\sum_{j=1}^{x} a_j^2 + a_{x+1}^2\right) + \left(2\sum_{j=1}^{x} a_j b_j + 2a_{x+1}b_{x+1}\right) + \left(\sum_{j=1}^{x} b_j^2 + b_{x+1}^2\right)$$

$$\left(\sum_{j=1}^{x} a_j^2 + a_{x+1}\right) + \left(2\sum_{j=1}^{x} a_j b_j + 2 a_{x+1} b_{x+1}\right)$$

$$+ \left(\sum_{j=1}^{x} b_j^2 + b_{x+1}\right)$$

$$= \sum_{j=1}^{x+1} a_j^2 + 2\sum_{j=1}^{x+1} a_j b_j + \sum_{j=1}^{x+1} b_j^2$$

3-3
ARITHMETIC PROGRESSION

① If each term of an arithmetic progression is multiplied by k, will the resulting terms form an arithmetic progression.

(a) if $k \neq 0$? YES
(b) if $k = 0$? YES

(a) When $k \neq 0$: If each term in the arithmetic progression is $a, a+d, a+2d, \ldots, a+(n-1)d$ is multiplied by k, the resulting sequence is $ka, ka+kd, ka+2kd, ka+3kd, \ldots, ka+(n-1)kd$. The second sequence has a common difference of $k \cdot d$.

(b) The resulting sequence is $0, 0, 0, 0, \ldots, 0$ with a common difference 0.

(2) If given the arithmetic mean of two numbers, can you find the two numbers?

No. If given $k = \frac{a+b}{2}$, we have no way of determining a and b.

(3) Can the sum of the terms of an arithmetic progression be zero? Explain.

Yes, if the first term is zero, and the common difference is zero, the sum of terms is zero.

if $2a + (n-1)d = 0$ or $a + [a + (n-1)d] = 0$, that is, if the sum of the first and the last terms is zero.

(4) If each term of an arithmetic progression is squared, will the resulting terms form an arithmetic progression?

(a) if $d = 0$? YES

(b) if $d \neq 0$? NO.

t_1^2 t_2^2 t_3^2

a^2 $(a+d)^2 = a^2 + 2ad + d^2$ $(a+2d)^2 = a^2 + 4ad + 4d^2$

No. Isn't this a geometric progression?

Explanations for 4.

(a) If $d=0$, the original arithmetic progression is

$a, a+0, a+2\cdot 0, a+3\cdot 0, \ldots, a+(n-1)\cdot 0,$

or $a, a, a, \ldots, a$.

Squaring each term results in $a^2, a^2, a^2, \ldots, a^2$ where $d=0$

(b) If $d \neq 0$. No, the original arithmetic progression is $a, a+d, a+2d, \ldots, a+(n-1)d$

The resulting sequence after squaring each term is $a^2, a^2+2ad+d^2, a^2+4ad+4d^2, \ldots a^2+2a(n-1)d+(n-1)^2d^2$, which has no common difference.

Find the indicated term in the given arithmetic progression.

(5) Eleventh term in $5, 8, 11, \ldots$

$a=5, \quad d=3, \quad n=11$

$t_n = a+(n-1)\cdot d$

$t_{11} = 5+(11-1)\cdot 3 = 5+30 = 35$

(6) Seventh term in $3\sqrt{2}, \sqrt{2}, -\sqrt{2}$

$a = 3\sqrt{2}, \quad d=-2\sqrt{2}, \quad n=7$

$t_7 = 3\sqrt{2} + (7-1)(-2\sqrt{2}) = 3\sqrt{2} - 12\sqrt{2} = -9\sqrt{2}$

Note: to find distance between $3\sqrt{2}$, $\sqrt{2}$, $-\sqrt{2}$

$\sqrt{2} - 3\sqrt{2} = -2\sqrt{2}$; also $-\sqrt{2} - \sqrt{2} = -2\sqrt{2}$

(7) Sixth term in $a-b,\ a,\ a+b$

$a = a-b$, $d = a-(a-b) = b$, $n = 6$

$t_6 = (a-b) + (6-1)b = a - b + 5b$
$= a + 4b$

(8) Tenth term in $\frac{3}{4}, \frac{13}{12}, \frac{17}{12}, \ldots$

$a = \frac{3}{4}$, $d = \frac{13}{12} - \frac{3}{4} = \frac{13}{12} - \frac{9}{12} = \frac{4}{12} = \frac{1}{3}$

$n = 10$

$t_{10} = \frac{3}{4} + (10-1)\cdot\frac{1}{3} = \frac{3}{4} + 3 = \frac{15}{4} = 3\frac{3}{4}$

Find the missing terms in the indicated arithmetic progression.

(9) $3,\ \underline{5},\ 7,\ \underline{9},\ \underline{11}$

(10) $2,\ \underline{\frac{17}{3}},\ \underline{\frac{28}{3}},\ 13,\ \underline{\frac{50}{3}}$

$a = 2$, $n = 4$, $t_4 = 13$
$13 = 2 + (4-1)d$
$11 = 3d$, $d = \frac{11}{3}$

(11) $\underline{13},\ 8,\ \underline{3},\ \underline{-2},\ -7$

$a = 8$, $n = 4$, $t_4 = -7$
$-7 = 8 + (4-1)d$
$-15 = 3d$, $d = -5$

(12) $3,\ \underline{\frac{31}{4}},\ \underline{\frac{25}{2}},\ \underline{\frac{69}{4}},\ 22$

$a = 3$, $n = 5$, $t_5 = 22$
$22 = 3 + 4d$
$19 = 4d$, $d = \frac{19}{4}$

(13) $\underline{3x},\ \underline{2x},\ x,\ \underline{0},\ -x$

(14) $a, \frac{a+b}{2}, b, \frac{3b-a}{2}, \frac{2b-a}{}$

$a = a,\ n = 3,\ t_3 = b$

$b = a + 2d$

$d = \frac{b-a}{2}$ so $a + \frac{b-a}{2} = \frac{2a+b-a}{2}$

$b + \frac{b-a}{2} = \frac{2b+b-a}{2} = \frac{3b-a}{2}$

$\frac{3b-a}{2} + \frac{b-a}{2} = \frac{4b-2a}{2} = \frac{2(2b-a)}{2}$

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(15) Which term of the arithmetic progression 2, 9, 16, ... is 142

$t_n = a + (n-1)d$

$a = 2,\ d = 7,\ t_n = 142$

$142 = 2 + (n-1) \cdot 7 = 2 + 7n - 7$

$142 = 7n - 5$

$7n = 147$

$n = 21$

142 is the 21^{st} term

(16) Which term of 8, 5, 2, ... is −28?

$a = 8$, $d = -3$, $t_n = -28$ $t_n = a + (n-1)\cdot d$

$\therefore -28 = 8 + (n-1)\cdot(-3) = 8 - 3n + 3 = 11 - 3n$

$3n = 39 \rightarrow n = 13$

(17) Find the sum of the first 18 terms of 4, 7, 10, ...

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ $a = 4$, $d = 3$, $n = 18$

$S_{18} = \frac{18}{2}\left[2\cdot 4 + 17(3)\right] = 9(8 + 51) = 531$

$\begin{array}{r} 590 \\ -59 \\ \hline 531 \end{array}$

(18) Find the sum of the first 30 positive integers.

$a = 1$, $d = 1$, $n = 30$

$S_{30} = \frac{30}{2}\left[2\cdot 1 + 29\cdot 1\right] = 15(31) = 465$

This conforms with $\sum_{i=1}^{30} n^2 = \frac{n(n+1)}{2} = 15(31) = 465$

(19) The first term of an arithmetic progression is 5. The seventeenth term is 53. Find the third term.

$a = 5$, $t_{17} = 53$, $n = 17$; $53 = 5 + 16d$

$t_3 = 5 + 2(3) = 11$ $48 = 16d \rightarrow d = 3$

(20) The common difference in an arithmetic progression is 3. The tenth term is 23. Find the first term.

$t_{10} = 23 = a + (10-1)\cdot 3 = a + 27$

$-4 = a$.

Find the sum of each series

(21) $\sum_{k=1}^{20}(3k-1) = 2+5+8+11+\dots+59$

Since $2S_n = n(a+t_n) \rightarrow S_n = \frac{n}{2}(a+t_n)$
and since $t_n = a+(n-1)d$

$$S_n = \frac{n}{2}\left[a+a+(n-1)\cdot d\right] = \frac{n}{2}\left[2a+(n-1)d\right]$$

So, rather than add twenty terms, I note the common difference is $d=3$, first term $a=2$, and $n=20$.

$$\therefore \quad S_{20} = \frac{20}{2}\left[2\cdot 2+(19)\cdot(3)\right] = 10(4+57) = 610$$

(22) $\sum_{i=1}^{10}(8-2i) = 6+4+2+\dots-12$

$a=6$, $d = 4-6 = -2$, $n=10$

$$S_{10} = \frac{10}{2}\left[2\cdot 6+(9)(-2)\right] = 5\cdot(12-18) = -30$$

(23) By mathematical induction, prove the theorem on page 81 (MIA) DOLCIANI

$\boxed{\text{THEOREM}}$ The sum of the first n terms of an arithmetic progression whose first term is a and whose common difference is d is:

$$S_n = \frac{n}{2}\left[2a + (n-1)\cdot d\right]$$

Let S be the set of integers n for which the theorem is true.

(1) $1 \in S$, since $S_1 = a = \frac{1}{2}\left[2\cdot a + (1-1)d\right]$

$= \frac{1}{2}(2\cdot a + 0) = a$

(2) Assume $x \in S$ and $S_x = \frac{x}{2}\left[2\cdot a + (x-1)\cdot d\right]$

Then $S_{x+1} = \frac{x+1}{2}\left[2\cdot a + (x+1-1)\cdot d\right] = \frac{x+1}{2}\left[2\cdot a + xd\right]$

and $x+1 \in S$; $S_x + t_{x+1} = S_x + (a + xd)$

$$= \frac{x}{2}\left[2\cdot a + (x-1)d\right] + (a + xd)$$

$$= \frac{2\cdot a\cdot x + x(x-1)\cdot d + 2a + 2xd}{2}$$

$$= \frac{(x+1)(2\cdot a) + (x-1+2)x\cdot d}{2} = \frac{(x+1)(2\cdot a) + (x+1)x\cdot d}{2}$$

$$= \frac{(x+1)(2a + xd)}{2} = \frac{x+1}{2}\left[2a + xd\right]$$

Thus, $S = \mathbb{N}$

(24) How many terms of the arithmetic progression $-5, -1, 3, \ldots$ must be added to give a sum of 400?

$d = -1-(-5) = 4$
$= 3-(-1)$

$$S_n = \frac{n}{2}\left[2\cdot a + (n-1)\cdot d\right]$$

$$400 = \frac{n}{2}\left[2(-5)+(n-1)(4)\right] = \frac{n}{2}\left[-10+4n-4\right]$$

$$800 = n(4n-14) = 4n^2 - 14n$$

$$2n^2 - 7n - 400 = 0$$

Factor $2x^2 - 7x - 400$ $\quad a=2,\ b=-7,\ c=-400$

$u\cdot v = a\cdot c = -800$, $u+v = b = -7$

FACTORS OF -800: $(1,800), (2,400), (4,200), (5,160), (8,100), (16,50), \boxed{(32,25)}$ $\quad (-32)\cdot(25) = -800$

$-32+25 = -7$

$$ax^2 + ux + vx + c \longrightarrow 2x^2 - 32x + 25x - 400$$

$$2x(x-16) + 25(x-16) = (2x+25)(x-16)$$

$$2n^2 - 7n - 400 = (2n+25)(n-16) = 0$$

when $2n+25=0$ or $n=16$

$2n = -25$

$n = \frac{-25}{2}$, which makes no sense, so $n=16$.

(25)

(26)

(25) Find the sum of all positive integers less than 200 that are multiples of 7.

let first term $a = 7$, common difference $d = 7$

4 $7 \cdot 28 = 196$, so 196 is last of 28 terms, that is, $n = 28$

$$S_n = \frac{n}{2}\left[2 \cdot a + (n-1)d\right]$$

$$S_{28} = \frac{28}{2}\left[2 \cdot 7 + (27)(7)\right] = 14 \cdot 29 \cdot 7 = 2842$$

(26) Find the first two terms of an arithmeter progression in which the third term is 14 and the ninth term is -1.

To find distance d, let third term be first term $a = 14$ and ninth term be seventh term: $t_7 = -1$

$$t_n = a + (n-1)d \;:\; t_7 = -1 = 14 + 6d$$

$$-15 = 6d \rightarrow d = -\frac{15}{6} = -\frac{5}{2} \qquad d = -\frac{5}{2}$$

$$t_2 = 14 - \left(\frac{5}{2}\right) = \frac{28}{2} + \frac{5}{2} = \frac{33}{2} = 16\tfrac{1}{2}$$

$$t_1 = \frac{33}{2} - \left(\frac{5}{2}\right) = \frac{38}{2} = 19$$

Another, more straight forward and direct approach:

$$t_3 = 14 = a + (3-1)\left(-\frac{5}{2}\right) = a + (-5)$$

$$19 = a, \text{ the first term}$$

Then $t_2 = 19 + (2-1)(-5/2) = 19 - \frac{5}{2} = \frac{38}{2} - \frac{5}{2} = \frac{33}{2}$

$= 16\frac{1}{2}$

(27) For what values of t will $t-2$, $2t-6$, and $4t-8$, in this order, form an arithmetic progression?

When $(2t-6)-(t-2) = (4t-8)-(2t-6)$

$$2t-6-t+2 = 4t-8-2t+6$$

$$t-4 = 2t-2$$

$$-2 = t \quad : -4, -10, -16$$

(28) For what real values of k (if any) will -1 be the arithmetic mean of k and k^2

when $k=-1$, $k^2=1$: $-1, 0, 1$ but $\frac{-1+1}{2}=0 \neq -1$

when $k=2$, $k^2=4$; $2, 3, 4$ $\quad \frac{2+4}{2}=3 \neq -1$

How ? $\quad k, \_, k^2$

$$t_3 = k^2 = k+(3-1)d = k+2d$$

$$k^2-k-2d=0$$

$$\left(k^2-k+\left(-\tfrac{1}{2}\right)^2\right) = 2d+\frac{1}{4}$$

$$\left(k-\tfrac{1}{2}\right)^2 = \frac{8d+1}{4}$$

$$k-\frac{1}{2} = \pm\frac{\sqrt{8d+1}}{2}$$

$$k = \frac{\pm\sqrt{8d+1}+1}{2}$$

$$k = \frac{1+\sqrt{8d+1}}{2}$$

or

$$k = \frac{1-\sqrt{8d+1}}{2}$$

if $d=1$; $k = \frac{1+3}{2} = 2$ or $k = \frac{1-3}{2} = -1$.

$$-1 = \frac{k+k^2}{2} \rightarrow -2 = k+k^2$$

$$k^2+k+2=0$$

$$k^2+k+\left(\tfrac{1}{2}\right)^2 = -2$$

$$\left(k+\tfrac{1}{2}\right)^2 = -2+\frac{1}{4} = -$$

(29)

Since $\left(k+\frac{1}{2}\right)^2 = -\frac{7}{4}$

$k+\frac{1}{2} = \sqrt{-7/4}$ which is NOT real, $\left(\frac{\sqrt{7}i}{2}\right)$.

So there are no real values of k where -1 is the arithmetic mean of k and k^2

The key is trying to solve $-1 = \frac{k+k^2}{2}$

C

(29) Find three numbers in an arithmetic progression whose sum is 27 and whose product is 288.

$t_1 = a$, $t_2 = a+d$, $t_3 = a+2d$

$a + (a+d) + (a+2d) = 27$

$3a + 3d = 27$

$3(a+d) = 27 \rightarrow a+d = 9 \therefore a = 9-d$

$(9-d) + (9-d+d) + (9-d+2d) = 27$

$(9-d) + 9 + (9+d) = 27$

if $d=1$: $8+9+10 = 27$, but $8 \cdot 9 \cdot 10 = 720$.

USE MATHEMATICAL LOGIC: $a = 9-d$.

$a(a+d)(a+2d) = 288$

$(9-d)(9)(9-d+2d) = (9-d)(9)(9+d) = 288$

$(9-d)(9+d) = 32$ FACTORS OF 32: $1 \cdot 32$, $\boxed{2 \cdot 16}$, $4 \cdot 8$

$(9-7)(9+7) = 2 \cdot 16 = 32$ WHEN $d=7$

$9-7=2$, 9, $9+7=16$: $2+9+16 = 27$

$2 \cdot 9 \cdot 16 = 288$

Note: When I reached $(9-d)(9+d) = 32$, I used "brute force" by testing factors of 32.

That is ok. It shows me my brain is trying to be "creative".

Another way would have been to expand:

$$(9-d)(9+d) = 81 - d^2 = 32$$

$$d^2 = 49 \longrightarrow d = 7.$$

---

(30) The first four terms of an arithmetic progression are r, s, t, u. Show that $r + u - s = t$.

$r = a$, $s = a + d$, $t = a + 2d$, $u = a + 3d$

$$r + u - s = a + (a + 3d) - (a + d) = (2a - a) + (3d - d)$$

$$= a + 2d = t$$

---

(31) The average of the n terms $a_1, a_2, a_3, \ldots, a_n$ is $\frac{1}{n}(a_1 + a_2 + a_3 + \ldots + a_n)$.

Show that if these terms are in arithmetic progression the average of the first and last terms is the same as the average of all n terms.

$a_1 = a_1$, $a_2 = a_1 + d$, $a_3 = a_1 + 2d, \ldots, a_n = a_1 + n \cdot d$

$$\frac{a_1 + a_1 + n d}{2} = \frac{2a_1 + n \cdot d}{2} = a_1 + \frac{n}{2} \cdot d$$

$a_2 = a_1 + d$, $a_3 = a_1 + 2d$, $a_4 = a_1 + 3d, \ldots, a_n = a_1 + (n-1)d$

32.

$\boxed{\text{Average of } a_1 \text{ and } a_n}$ $\frac{1}{2}(a_1 + a_n) = \frac{1}{2}(a_1 + (a_1 + (n-1)d)) = \frac{1}{2}(2a_1 + (n-1)d)$

$\boxed{\text{Total Average}}$ $\frac{1}{n}(a_1 + a_2 + a_3 + \ldots + a_n) = \frac{1}{n}[a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)] = \frac{1}{n} S_n$

$S_n = \frac{n}{2}(2a + (n-1)d) \therefore \frac{1}{n} S_n = \frac{1}{n} \cdot \frac{n}{2}(2a + (n-1)d)$

$= \frac{1}{2}(2a + (n-1)d) = \frac{1}{2}(a_1 + a_n)$

---

(32) Show that $S_n = \sum_{k=1}^{n} [a + (k-1) \cdot d]$ and then use the theorems stated in Exercises 47, 48, and 49, (page 79 MIA), and in exercise 14, page 73, to prove that $S_n = \frac{n}{2}[2a + (n-1)d]$.

$\boxed{3\text{-}1 \ \#14}$ $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

$\boxed{3\text{-}2 \ \#47}$ $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$\boxed{3\text{-}2 \ \#48}$ $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

$\boxed{3\text{-}2 \ \#49}$ $\sum_{i=1}^{n} a = a \cdot n$

Note: $0 + 1 + 2 + \ldots + (n-1) = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)(n)}{2}$

First show that $S_n = \sum_{k=1}^{n} (a+(k-1)d)$.

$$S_n = (a+0\cdot d) + (a+d) + (a+2d) + \dots + (a+(n-1)d)$$

$$= \sum_{k=1}^{n} (a+(k-1)d).$$

$$\sum_{k=1}^{n} (a+(k-1)d) = \sum_{k=1}^{n} a + \sum_{k=1}^{n} (k-1)\cdot d$$

$$= \sum_{k=1}^{n} a + d\sum_{k=1}^{n} (k-1) = a\cdot n + d \sum_{k=1}^{n} (k-1)$$

$$= a\cdot n + d\,\frac{(n-1)(n)}{2} = \frac{2\cdot a\cdot n + d\,(n-1)(n)}{2}$$

$$= \frac{n}{2}\left[2\cdot a + (n-1)d\right]$$

8k

---

(33) A woodsman stacks $8k+15$ logs in such a way that there are k layers with 8 logs in the top layer. Each layer below contains one more log than the layer immediately above. Find the number of logs.

This is in arithmetic progression with the first term $a = 8$, the common difference $d = 1$

$$S_n = 8 + (8+1) + (8+2) + \dots + (8 + (n-1))$$

$$1+2+3+4+5 = 15$$

There are 6 layers, so there are $8(6) + 15 = 63$ logs.

This was "brute force". Is there a more "elegant" or "mathematically sophisticated" method? Of course there is.

$a = 8$, $d = 1$, $n = k$, and $S_n = 8k + 15$

$$S_n = 8k + 15 = \frac{k}{2}\left[2 \cdot a + (k-1)\,d\right]$$

$$= \frac{k}{2}\left[2(8) + (k-1)(1)\right] = \frac{k}{2}(16 + k - 1) = \frac{k}{2}(15 + k)$$

$$8k + 15 = \frac{15k}{2} + \frac{k^2}{2} \longrightarrow 16k + 30 = 15k + k^2$$

$$k^2 - k - 30 = 0 \longrightarrow \left(k^2 - k + \left(-\tfrac{1}{2}\right)^2\right) = 30 + \frac{1}{4}$$

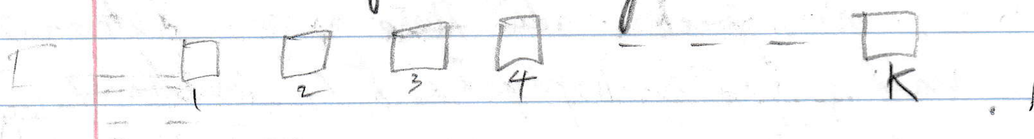
$$\left(K - \frac{1}{2}\right)^2 = \frac{120}{4} + \frac{1}{4} = \frac{121}{4},$$

$$K - \frac{1}{2} = \pm\frac{11}{2} \longrightarrow \begin{cases} K = \frac{11}{2} + \frac{1}{2} = 6 \\ K = -\frac{11}{12} + \frac{1}{2} = -5 \end{cases}$$

Since $K > 0$, $k = 6$ $\therefore$ $8k + 15 = 8 \cdot 6 + 15 = 63$

The second method is purely analytical without any thoughts about logs, whereas with the brute force method, I was visualizing the layers, adding one log to 8, then 2, then 3, etc...

(34) There are k animal feeding stations arranged in a line with a supply hut. An attendant carries n bags of feed, one at a time, to each feeding station. How far will he have traveled when he arrives back at the hut after servicing all stations?



$a = n(2t)$, $d = n(2s)$

number of trips $= k$

total distance $= \frac{k}{2}\left[2(2 \cdot n \cdot t) + (k-1)(2n \cdot s)\right]$

$= k\left[2 \cdot n \cdot t + (k-1)(n \cdot s)\right]$

$= k \cdot n\left[2t + (k-1)s\right]$

3-4 GEOMETRIC PROGRESSIONS

① Can a term of a geometric progression be zero? Explain your answer. Yes. If $a = 0$ or $r = 0$.

② By how much does the arithmetic mean of 4 and 9 exceed the absolute value of their geometric mean?

The arithmetic mean is $\frac{4+9}{2} = \frac{13}{2}$.

The geometric mean is ?

$t_n = a \cdot r^{n-1}$; $a = 4$, $n = 3$, $t_3 = 9$

$t_3 = 9 = 4(r)^{3-1} = 4r^2 \rightarrow r^2 = \frac{9}{4}$

$r = \pm\frac{3}{2}$ so $r = \frac{3}{2}$ or $r = -\frac{3}{2}$

$4 \cdot \frac{3}{2} = 2 \cdot 3 = 6$ or $4 \cdot \left(-\frac{3}{2}\right) = -6$

$6 \cdot \frac{3}{2} = 9$ or $-6 \cdot \left(-\frac{3}{2}\right) = 9$

The geometric mean is 6 or -6, so its absolute value (magnitude) is 6. $\frac{13}{2} - \frac{12}{2} = \frac{1}{2}$

③ Find t_n in the geometric progression in which $a = 12$, $n = 5$, and $r = \frac{1}{3}$.

$t_5 = (12)\left(\frac{1}{3}\right)^4 = 12\left(\frac{1}{81}\right) = \frac{4}{27}$

④ Insert three real geometric means between $\frac{27}{8}$ and $\frac{2}{3}$.

$t_5 = \frac{2}{3} = \left(\frac{27}{8}\right)r^4 \rightarrow r^4 = \frac{16}{81} \rightarrow r^4 - \frac{16}{81} = 0$

$\left(r^2 - \frac{4}{9}\right)\left(r^2 + \frac{4}{9}\right) = 0 \rightarrow r^2 = \frac{4}{9}$ so $r = \pm\frac{2}{3}$

Since common ratio r is $\frac{2}{3}$ or $-\frac{2}{3}$ and the first and fifth terms are $\frac{27}{8}$ and $\frac{2}{3}$, respectively,

$$\frac{27}{8},\ \frac{54}{24}=\frac{9}{4},\ \frac{18}{12}=\frac{3}{2},\ 1,\ \frac{2}{3}$$

or $$\frac{27}{8},\ -\frac{9}{4},\ \frac{3}{2},\ -1,\ \frac{2}{3}$$

---

(5) Find the seventh term in the geometric progression

$$\frac{3}{64},\ -\frac{3}{16},\ \frac{3}{4},\ -3,\ \ldots$$

$$a=\frac{3}{64},\quad t_4=-3=\left(\frac{3}{64}\right)(r)^3$$

$$r^3=-64 \rightarrow r^3=(-4)^3 \quad \therefore r=-4$$

$$t_7=\left(\frac{3}{64}\right)(-4)^6=\frac{3}{64}\cdot 4096=3\cdot 64=192$$

---

(6) Find the sum of 100 terms of the geometric series

$1-1+1-1+\ldots$

$$S_n=\frac{a-a\cdot r^n}{1-r} \rightarrow S_{100}=\frac{1-1\cdot(-1)^{100}}{1-(-1)}=\frac{1-1}{2}=0$$

(7) In a geometric progression containing only real terms, the first term is 3 and the fourth term 24. Find the common ratio. $t_4=24=(3)r^3$

$r^3=8 \quad \therefore r=2$

(8) The first term of a geometric progression is 4 and the last term is 324. Find the sum if the ratio of two successive terms is 3.

$t_n = 324 = 4\cdot(3)^{n-1} \longrightarrow 81 = 3^{n-1} = 3^4$

$\therefore\ n-1 = 4 \rightarrow n = 5$

$S_5 = \dfrac{4-4\cdot 3^5}{1-3} = \dfrac{4-4\cdot 243}{-2} = \dfrac{4-972}{-2} = \dfrac{-968}{-2} = 484$

$t_n = a\cdot r^{n-1}$ so $a\cdot r^n = t_n r$

$S_n = \dfrac{a-t_n r}{1-r} = \dfrac{4-3\cdot 324}{1-3} = \dfrac{-968}{-2} = 484$

{9,..12} Find the sum of the indicated geometric series.

(9) $\displaystyle\sum_{k=1}^{5} 2^{k-1} = 2^0+2^1+2^2+2^3+2^4 = 1+2+4+8+16$

$= 31$

Using $S_n = \dfrac{a-a\cdot r^n}{1-r}$, $a=1$, $r=2$, $n=5$

$S_5 = \dfrac{1-1\cdot 2^5}{1-2} = \dfrac{-31}{-1} = 31$ ✓

(10) $\displaystyle\sum_{j=1}^{3} 5\cdot(4)^{j-1} = 5\cdot 4^0 + 5\cdot 4 + 5\cdot 4^2 = 5+20+80 = 105$

Using $S_n = \dfrac{a-a\cdot r^n}{1-r}$, $a=5$, $r=4$, $n=3$

$S_3 = \dfrac{5-5\cdot 4^3}{1-4} = \dfrac{5-320}{-3} = \dfrac{-315}{-3} = 105$ ✓

$\begin{array}{r} \overset{2}{64} \\ \times\ 5 \\ \hline 320 \end{array}$

(11) $$\sum_{i=1}^{5} 3\left(-\frac{1}{3}\right)^{i-1} = 3\cdot\left(-\frac{1}{3}\right)^0 + 3\cdot\left(-\frac{1}{3}\right)^1 + 3\left(-\frac{1}{3}\right)^2 + 3\cdot\left(-\frac{1}{3}\right)^3$$

$$+ 3\cdot\left(-\frac{1}{3}\right)^4 = 3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27}$$

$$= \frac{81}{27} - \frac{27}{27} + \frac{9}{27} - \frac{3}{27} + \frac{1}{27} = \frac{91}{27} - \frac{30}{27} = \frac{61}{27}$$

Using $S_n = \dfrac{a - a\cdot r^n}{1-r}$, where $a = 3$, $r = \left(-\frac{1}{3}\right)$, $n = 5$

$$S_5 = \frac{3 - 3\cdot\left(-\frac{1}{3}\right)^5}{1-\left(-\frac{1}{3}\right)} = \frac{3 - 3\cdot\left(-\frac{1}{243}\right)}{1+\frac{1}{3}} = \frac{3 - \left(-\frac{1}{81}\right)}{\frac{4}{3}}$$

$$= \frac{3}{4}\left[\frac{243}{81} + \frac{1}{81}\right] = \frac{244}{4\cdot 27} = \frac{61}{27} \quad ✓$$

(12) $$\sum_{r=1}^{6} 24\left(\frac{1}{2}\right)^{r-1} \qquad a = 24,\ r = \frac{1}{2},\ n = 6$$

$$S_6 = \frac{24 - 24\cdot\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = \frac{24 - 24\cdot\left(\frac{1}{64}\right)}{\frac{1}{2}}$$

$$= 2\left(24 - \frac{3}{8}\right) = 2\left(\frac{192}{8} - \frac{3}{8}\right) = 2\left(\frac{189}{8}\right) = \frac{189}{4}$$

(13) Find the seventh term of a geometric progression whose third and fifth terms are respectively $\frac{9}{4}$ and $\frac{81}{64}$.

Solution: To find the common ratio r, I let $t'_1 = \frac{9}{4}$ and $t'_3 = \frac{81}{64} = \left(\frac{9}{4}\right) r^{3-1} = \frac{9}{4} r^2$

$$\frac{4}{9}\cdot\frac{81}{64} = \frac{9}{16} = r^2 \text{ so } r = \frac{3}{4} \text{ or } r = -\frac{3}{4}$$

Now one needs to know what t_1 is... because we don't know "a".

$$t_3 = \frac{9}{4} = a \cdot r^{3-1} = a \cdot \left(\frac{3}{4}\right)^2 = \frac{9}{16} \cdot a$$

$$a = \frac{9}{4} \cdot \frac{16}{9} = 4 \quad \therefore \quad t_7 = 4 \cdot \left(\frac{3}{4}\right)^6 = 4 \cdot \left(\frac{729}{4096}\right) = \frac{729}{1024}$$

Actually, to find r I could have just used $t_3 = \frac{9}{4} = ar^2$

Then $t_5 = \frac{81}{64} = a \cdot r^4 = a \cdot r^2 (r^2) = t_3 r^2 = \frac{9}{4} r^2$

$r^2 = \frac{4}{9} \cdot \frac{81}{64} = \frac{9}{16} \quad \therefore \; r = \frac{3}{4}$ or $r = -\frac{3}{4}$ as before.

Keep this technique in mind.

(14) Under conditions favorable to the growth of a certain bacteria, one organism can divide into two every half hour. How many times the original number of organisms will there be at the end of a six hour period?

Brute force answer: 4096. There are 12 doublings in 6 hours. $2^{12} = 4096$.

---

alternately, 2 after ½ hour, where $a = 1$ *

$r = 2$

$n = 13$

$$t_{13} = a \cdot r^{13-1} = 1 \cdot 2^{12} = 4096$$

Notice that $t_{n+1} = t_n r$ and $t_n = ar^{n-1}$

$$t_{n+1} = a \cdot r^n$$

(15) For what values of k will $k-4$, $k-1$, and $2k-2$, in this order, form a geometric progression?

when $$\frac{k-1}{k-4} = \frac{2k-2}{k-1} = \frac{2(k-1)}{k-1} = 2 \qquad k \neq 4, 1$$

$$k-1 = 2(k-4) = 2k-8$$

$$k = 7$$

Hence, when $k=7$: $7-4,\ 7-1,\ 2(7)-2$

$3,\ 6,\ 12$ with common ratio 2

Also, if $k=1$, this is a geometric progression $-3, 0, 0$ with common ratio 0.

(16) Solve $\sum_{k=1}^{n} 2^k = 62$ for n by testing successive values of n.

$$\sum_{k=1}^{5} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$

$$= 2 + 4 + 8 + 16 + 32 = 62$$

$\therefore\ n = 5$

The statements in exercises 17 and 18 refer to a geometric progression whose first term is a, whose common ratio is r, and whose n^{th} term is t_n. Prove each statement by mathematical induction.

(17) $t_n = a \cdot r^{n-1},\ n \geq 1$

(1) $1 \in S$: $t_1 = a \cdot r^0 = a,\ r \neq 0$

(2) Assume $k \in S$ and $t_k = a \cdot r^{k-1}$

2 Then $t_{k+1} = a \cdot r^{k+1-1} = a \cdot r^k$ and $k+1 \in S$:

By definition $t_{k+1} = t_k \cdot r = a \cdot r^{k-1} \cdot r = a \cdot r^k \therefore S = \mathbb{N}$

(18) $$\sum_{k=1}^{n} t_k = \frac{a(1-r^n)}{1-r},\ r \neq 1$$

Let S be the set of integers $n > 0$ for which the statement is true.

(1) $1 \in S$: $$\sum_{k=1}^{1} t_k = \frac{a(1-r^1)}{1-r} = \frac{a - a \cdot r}{1-r}$$

(2) Assume $x \in S$ and $\sum_{k=1}^{x} t_k = \frac{a(1-r^x)}{1-r}$

Then $$\sum_{k=1}^{x+1} t_k = \frac{a(1-r^{x+1})}{1-r}$$ and $x+1 \in S$:

$$\sum_{k=1}^{x+1} t_k = \sum_{k=1}^{x} t_k + \underbrace{a \cdot r^x}_{t_{x+1} = a \cdot r^x}$$

$$= \frac{a(1-r^x)}{1-r} + a \cdot r^x = \frac{a(1-r^x) + a \cdot r^x(1-r)}{1-r}$$

$$= \frac{a - a \cdot r^x + a \cdot r^x - a \cdot r^{x+1}}{1-r} = \frac{a - a \cdot r^{x+1}}{1-r}$$

$$= \frac{a(1-r^{x+1})}{1-r}$$

{19, , 24} Three of the five real numbers, a, t_n, r, n, and S_n for a geometric progression are given. Find the two numbers not given.

(19) $a = 1$, $r = 2$, $n = 7$

$$t_7 = a \cdot r^6 = 1 \cdot 2^6 = 64$$

$$S_n = \frac{a - t_n r}{1-r} = \frac{a - a \cdot r^n}{1-r} \quad \text{because } t_n = a \cdot r^{n-1}$$

$$S_7 = \frac{1 - 64 \cdot 2}{1-2} = \frac{1-128}{-1} = 127$$

20 $a = -1,\ n = 3,\ S_n = -7$

$$S_n = \frac{a - a \cdot r^n}{1-r} \rightarrow S_3 = -7 = \frac{-1 - (-1) \cdot r^3}{1-r}$$

$$-7 = \frac{-1 + r^3}{1-r} = \frac{(r-1)(r^2+r+1)}{1-r}$$

$$-7 = \frac{-(1-r)(r^2+r+1)}{1-r} = -r^2 - r - 1$$

note:

$(r-1)(r^2+r+1) = r^3 + r^2 + r - r^2 - r - 1 = r^3 - 1$

and $-(1-r) = r-1$

$r^2 + r - 6 = 0 \rightarrow (r+3)(r-2) = 0 \rightarrow r = -3$ or $r = 2$

$t_3 = (-1)(-3)^2 = -9$ or $t_3 = (-1)2^2 = -4$

21 $a = -2,\ r = 2,\ t_n = -64$

$$t_n = -64 = (-2)(2)^{n-1} \rightarrow \frac{-64}{-2} = 2^{n-1} \rightarrow 32 = 2^{n-1}$$

$$2^5 = 2^{n-1} \therefore n - 1 = 5 \rightarrow \boxed{n = 6}$$

$$S_6 = \frac{a - t_n r}{1-r} = \frac{(-2) - (-64)(2)}{1-2} = \frac{-2+128}{-1} = -126$$

also $$S_6 = \frac{a - a \cdot r^n}{1-r} = \frac{-2 - (-2)(2^6)}{1-2} = \frac{-2+128}{-1} = -126$$

(22) $r = \frac{1}{2}, \; n = 5, \; S_n = \frac{31}{2}$

$$S_5 = \frac{31}{2} = \frac{a - a\cdot\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}} = \frac{a\left(1-\frac{1}{32}\right)}{\frac{1}{2}}$$

$$a\left(\frac{31}{32}\right) = \frac{31}{2}\cdot\frac{1}{2} = \frac{31}{4}$$

$$a = \frac{31}{4}\cdot\frac{32}{31} = 8$$

$$t_n = a\cdot r^{n-1} = 8\cdot\left(\frac{1}{2}\right)^4 = t_5 = 8\cdot\left(\frac{1}{16}\right) = \frac{1}{2}$$

(23) $a = \frac{1}{3}, \; r = 3, \; S_n = \frac{40}{3}$

$$S_n = \frac{40}{3} = \frac{\frac{1}{3} - \left(\frac{1}{3}\right)(3)^n}{1-3} = \frac{\frac{1}{3} - 3^{-1}\cdot 3^n}{-2} = \frac{\frac{1}{3} - 3^{n-1}}{-2}$$

$$-\frac{80}{3} = \frac{1}{3} - 3^{n-1} \rightarrow -\frac{81}{3} = -3^{n-1}$$

$$\frac{81}{3} = 3^{n-1} \rightarrow 81 = 3^n \rightarrow 3^4 = 3^n$$

$$\therefore n = 4$$

$$t_4 = a\cdot r^{n-1} = \left(\frac{1}{3}\right)(3)^3 = 3^2 = 9$$

(24) $a = \frac{4}{3}, \; r = \frac{3}{2}, \; t_n = \frac{27}{4}$

$$t_n = \frac{27}{4} = \left(\frac{4}{3}\right)\left(\frac{3}{2}\right)^{n-1}$$

$$\frac{3}{4}\cdot\frac{27}{4} = \frac{81}{16} = \left(\frac{3}{2}\right)^{n-1} \rightarrow \left(\frac{3}{2}\right)^4 = \left(\frac{3}{2}\right)^{n-1}$$

$\therefore\ n-1=4 \rightarrow n=5$ so $t_5=\frac{27}{4}$

$$S_5=\frac{a-t_5\cdot r}{1-r}=\frac{\frac{4}{3}-\frac{27}{4}\cdot\frac{3}{2}}{1-\frac{3}{2}}=\frac{\frac{4}{3}-\frac{81}{8}}{-\frac{1}{2}}=\frac{\frac{32-243}{24}}{-\frac{1}{2}}$$

$$=-2\left(\frac{-211}{24}\right)=\frac{211}{12}\ ;\ \text{Also } S_5=\frac{a-a\cdot r^n}{1-r}=\frac{\frac{4}{3}-\frac{4}{3}\cdot\left(\frac{3}{2}\right)^5}{1-\frac{3}{2}}$$

$$=\frac{\frac{4}{3}-\frac{4}{3}\cdot\left(\frac{243}{32}\right)}{-\frac{1}{2}}=-2\left(\frac{4}{3}-\frac{81}{8}\right)=-2\left(\frac{32-243}{24}\right)$$

$$=-2\left(\frac{-211}{24}\right)=\frac{211}{12}\ \checkmark$$

(25) Insert two real numbers between 2 and 9 so that the first 3 terms form an arithmetic progression and the last 3 terms form a geometric progression.

$2, \_\_, \_\_, 9$

$\boxed{\text{Problem for the "Suicide Club"}}$

I figured it out with "brute mental force" : 2, 4, 6, 9 because 2, 4, 6 is an AP with common difference 2 and 4, 6, 9 is a GP with common ratio $\frac{3}{2}$

How about a more sophisticated method ?

For the AP we have : $t_1=2,\ t_2=2+d,\ t_3=2+2d=2(1+d)$

For the GP we have : $t_2=2+d,\ t_3=(2+d)r,\ t_4=(2+d)r^2$

also, $t_4=9=(2+d)\cdot r^2$

Thus $t_3=2(1+d)=(2+d)r\ \therefore\ r=\frac{2(1+d)}{2+d}$

$$t_4=9=(2+d)\left[\frac{2(1+d)}{2+d}\right]^2=\frac{[2(1+d)]^2}{2+d}$$

So now we have $t_4 = 9 = \frac{[2(1+d)]^2}{2+d}$

We note that $d \neq -2$ since then $t_2 = 2+d = 0$, and if 0 is the first term in a geometric progression, then all terms are 0. But $t_4 = 9$.

We have $r = \frac{2(1+d)}{2+d}$ and $t_4 = 9 = \frac{[2(1+d)]^2}{2+d}$

$$9(2+d) = [2(1+d)]^2 = (2+2d)^2 = 4+8d+4d^2$$
$$18+9d = 4+8d+4d^2$$
$$4d^2-d-14=0 \quad \text{Factor with computer or brain.}$$

$a \cdot c = u \cdot v = -56$: $(1,56), (2,28), (4,14), (8,7)$ ∴ $(-8)(7) = -56$

$b = -1 = -8+7$

$$4d^2-8d+7d-14 = 4d(d-2)+7(d-2)$$
$$\therefore 4d^2-d-14 = (4d+7)(d-2) = 0$$
when $d=2$ or $4d=-7 \rightarrow d = -\frac{7}{4}$

For $d=2$: $2, \underline{4}, \underline{6}, 9$

For $d = -\frac{7}{4}$: $2, \underline{\frac{1}{4}}, \underline{-\frac{3}{2}}, 9$

$d=2$:
$$r = \frac{2(1+2)}{2+2} = \frac{3}{2}$$

$d = -\frac{7}{4}$
$$r = \frac{2\left(1-\frac{7}{4}\right)}{2-\frac{7}{4}} = \frac{2\left(-\frac{3}{4}\right)}{\frac{1}{4}} = 4\left(-\frac{3}{2}\right) = -6$$

So, there were more than one solution.

(26) A car purchased for \$2500 depreciates 15% in value every year. Find the value of the car at the end of a four-year period.

\$1304.74 [brute force]

As a geometric progression, $a = 2500$, $r = 0.85$, $n = 5$

$t_5 = a \cdot r^{5-1} = 2500 \cdot (0.85)^4 = 1305.82$

$n \neq 4$ because $t_1 = 2500$, $t_2 = 2500 * 0.85$

Note also that $r \neq -0.15$ but $r = 1 - 0.15 = 0.85$

(27) Find the second term of an arithmetic progression whose first term is 2 and whose first, third, and seventh terms form a geometric progression.

* $t_1 = 2$

$t_2 = 2 + d$

* $t_3 = 2 + 2d = 2(1+d) = 2r$

$t_4 = 2 + 3d$

$t_5 = 2 + 4d$

$t_6 = 2 + 5d$

* $t_7 = 2 + 6d = (2 + 2d) + 4d = 2r^2 = 2r + 4d$

$r = \frac{t_3}{t_1} = \frac{t_7}{t_3} = \frac{2+6d}{2+2d} = \frac{2+2d}{2}$

$2(2+6d) = (2+2d)^2$

$4 + 12d = 4 + 8d + 4d^2$

$4d^2 - 4d = 0 \rightarrow 4d(1-d) = 0$

$d = 0$ or 1 $t_2 = 2$ or 3

(28) The sum of the first and last terms of a geometric progression of fourteen real terms is 7. The fifth term is the mean proportional between the second and last terms. Find the third term.

$t_1 = a$ $\qquad$ $a + a \cdot r^{13} = 7$

$t_n = a \cdot r^{n-1}$ $\qquad$ $a(1 + r^{13}) = 7$

$t_{14} = a \cdot r^{13}$ $\qquad$ $t_1 + t_{14} = 7$

$t_2 = a \cdot r$ $\qquad$ $a \leftarrow (a \cdot r)$ $\quad n \leftarrow 13, \; t_{13} = a \cdot r^{13}$

$t_5 = a \cdot r^4$ $\qquad$ $a \cdot r^{13} = (a \cdot r) r^{12}$

$\boxed{(t_5)^2 = (t_{14})(t_2)}$ ← mean proportional

$(a \cdot r^4)^2 = (a \cdot r^{13})(a \cdot r) = a^2 r^8 = a^2 r^{14}$

$r^8 = r^{14} \quad \therefore r = 1$ or $r = -1$

$\boxed{t_3 = a \cdot r^2}$ $\qquad$ If $r = 1$: $\quad a(1 - r^{13}) = 7$

$0 \neq 7$

so $\boxed{r = -1}$: $\quad a(1 - (-1)) = 7$

$a \cdot (2) = 7$

$\boxed{a = \frac{7}{2}}$

$t_3 = \left(\frac{7}{2}\right)(-1)^2 = \frac{7}{2}$

(29) One third of the air in a tank is removed with each stroke of a vacuum pump. What part of the original amount of air remains in the tank after five strokes?

$1-\frac{1}{3}=\frac{2}{3}$ ∴ $r=\frac{2}{3}$

t_1 Starts at a
t_2 After one stroke: $a\left(\frac{2}{3}\right)$
t_3 After two strokes: $a\left(\frac{2}{3}\right)^2$
t_4 After three strokes: $a\left(\frac{2}{3}\right)^3$
t_5 After four strokes: $a\left(\frac{2}{3}\right)^4$
t_6 After five strokes: $a\left(\frac{2}{3}\right)^5 = \frac{32}{243} \approx 13\%$

$t_6 = a \cdot r^5$

(30) Show that the reciprocals of the terms of a geometric progression $(a \neq 0,\ r \neq 0)$ form a geometric progression.

Let $a, ar, ar^2, \ldots, ar^{n-1}, \ldots$ denote a geometric series with common ratio r $(a \neq 0, r \neq 0)$.

Then the reciprocals of these terms are, $\frac{1}{a}, \frac{1}{a \cdot r}, \frac{1}{a \cdot r^2}, \ldots, \frac{1}{ar^{n-1}}, \ldots$ with common ratio $\frac{1}{r}$.

(31) Prove: If $a \in \mathbb{R}$ and $b \in \mathbb{R}$, the arithmetic mean of a^2 and b^2 are not less than the absolute value of their geometric mean.

[HINT: $(a-b)^2 \geq 0$]

Arithmetic mean is $\frac{a^2+b^2}{2}$

Their geometric mean is ab or $-ab$

Since all squares are positive, $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$\frac{a^2+b^2}{2} \geq a \cdot b$, and $(a+b)^2 \geq 0$

$a^2 + 2ab + b^2 \geq 0$.

$\frac{a^2+b^2}{2} \geq -ab$

Since $\frac{a^2+b^2}{2} \geq +ab$, $\frac{a^2+b^2}{2} \geq |ab|$

(32) For every sequence which is both an arithmetic progression and a geometric progression, determine the common (a) difference
(b) ratio.

(a)

$$\left.\begin{array}{l} t_1 = a \\ t_2 = a+d \\ t_3 = a+2d \\ t_n = a+(n-1)d \end{array}\right\}$$ denote an arithmetic progression and suppose that $a \neq 0$.

Then, since the sequence is also a geometric progression,

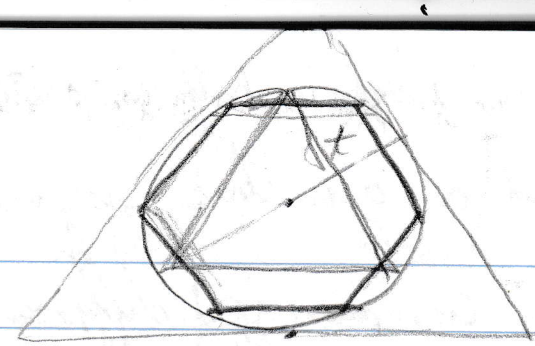
$$r = \frac{a+d}{a} = \frac{a+2d}{a+d} \quad \text{or} \quad r = 1 + \frac{d}{a} = 1 + \frac{d}{a+d}$$

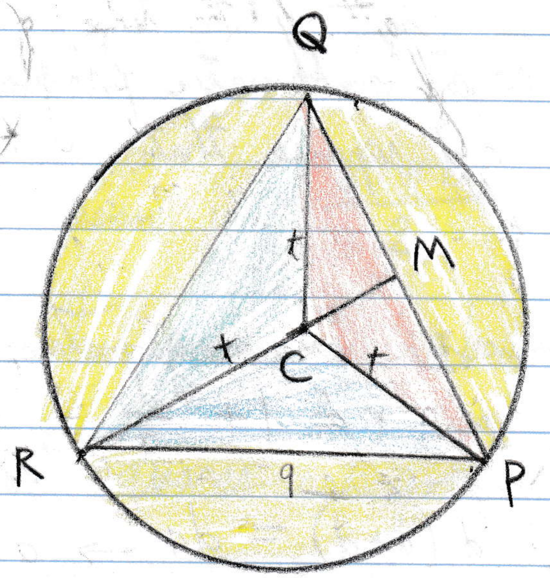
$$\frac{d}{a} = \frac{d}{a+d} \longrightarrow d(a+d) = da \rightarrow a = a+d,$$

$$\therefore d = 0$$

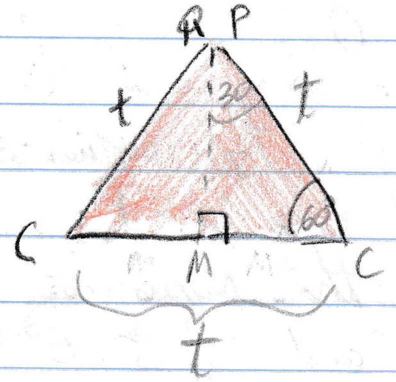
(b) If $a \neq 0$ and $d = 0$, the sequence becomes $a, a, a, a, \ldots$ and $r = 1$.
If $a = 0$, $d = 0$, the sequence becomes $0, 0, 0, 0, \ldots$ and $r \in \mathbb{R}$.

(33) Given a circle of radius t. Show that the area of the inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.





We can find the area of ΔPQC and multiply by 3 to find area of Δ PQR.



$\sin(60^\circ) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\overline{PC} = t$

$\overline{MC} = \frac{t}{2}$

$$|PM|^2 + |MC|^2 = |PC|^2$$

$$|PM|^2 = |PC|^2 - |MC|^2 = t^2 - \left(\frac{t}{2}\right)^2$$

$$\overline{PM} = \sqrt{t^2 - \frac{t^2}{4}} = \sqrt{\frac{3t^2}{4}} = \frac{t}{2} \cdot \sqrt{3}$$

area of $\Delta PCQ = \frac{1}{2} \cdot t \cdot \frac{\sqrt{3}}{2} t = \frac{\sqrt{3}\, t^2}{4}$

So, the area of the inscribed equilateral triangle is

$$\boxed{\frac{3}{4}\cdot t^2\sqrt{3} = A_I}$$

Another way to tackle this first part of the problem is to find what the length of the sides of the inscribed equilateral triangle is directly. $\sqrt{3}t$.

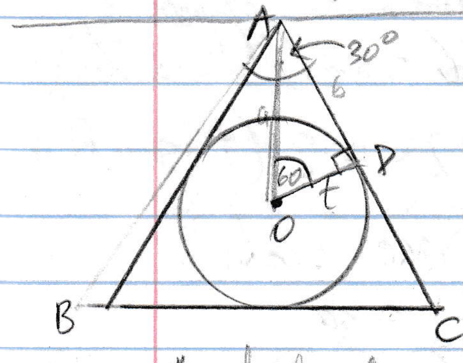
Applying the law of sine to the triangle ΔRCP, we get $\frac{q}{\sin(60^\circ)} = \frac{t}{\sin(30^\circ)} \rightarrow q = t\cdot\frac{\sin(60^\circ)}{\sin(30^\circ)}$

$$q = t\cdot\left(\frac{\sqrt{3}/2}{1/2}\right) = \sqrt{3}\cdot t$$

So the length of the sides is each $\sqrt{3}\,t$

The area of ΔRQP is $\frac{1}{2}(\sqrt{3}t)(t+\frac{t}{2}) = \frac{1}{2}(\sqrt{3}t)(\frac{3t}{2})$
$= \frac{3}{4}t^2\sqrt{3}$, confirming the first result.

$\left(\frac{t}{\sin(30)} = \frac{q}{\sin(90)} \rightarrow \frac{t}{1/2} = \frac{q}{1}\right.$



radius t

ΔOAD is 30-60-90, so the sides have ratio of $1 : \sqrt{3} : 2$

$\overline{OD} = t$, $\overline{OA} = 2t$, $\overline{AD} = \sqrt{3}t$

$\therefore \overline{AC} = 2\cdot t\sqrt{3} = [2\cdot\overline{AD}]$

The height of the triangle is $(2t+t) = 3t$ $\therefore$ Area $= \frac{1}{2}(3t)(2t\sqrt{3})$
$= 3t^2\sqrt{3}$

$\frac{t}{\sin(30)} = \frac{b}{\sin(60)}$

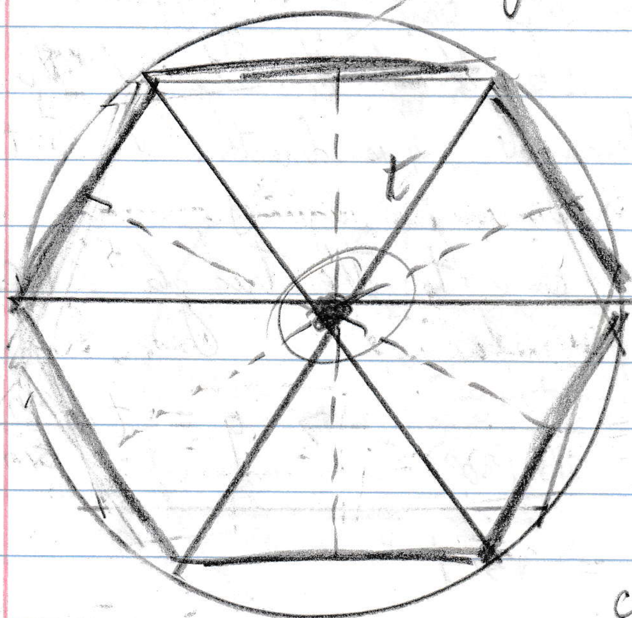
$2t = \frac{b}{\sqrt{3}/2} \rightarrow b = \sqrt{3}\cdot t$

$$\boxed{A_c = 3t^2\sqrt{3}}$$

So, so far, the area of inscribed triangle: $\frac{3}{4}t^2\sqrt{3} = A_I$

and area of circumscribed triangle: $3t^2\cdot\sqrt{3} = A_C$

Now, the inscribed hexagon.

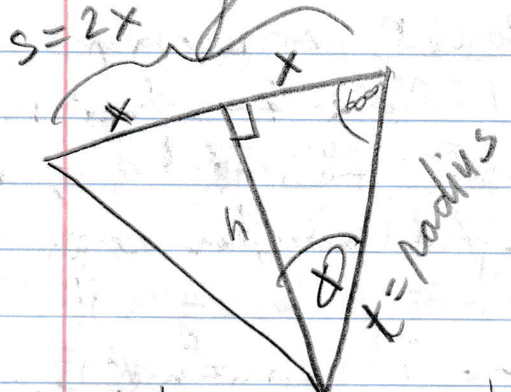


The radii to the 6 corners of the inscribed hexagon creates 6 isosceles triangles.

Drawing 6 bisectors, we have 12 congruent right triangles

The side length of the hexagon is two of the short sides of the right triangle.

The short side of the right triangle is opposite the angle at the circle's center



Thus $s = 2x = 2(t\cdot\sin\theta)$

Based on 360° circle:

$\theta = \left(\frac{360}{12}\right)^\circ = 30^\circ$

$s = 2x = 2(t\sin(30)) = 2\cdot t\left(\frac{1}{2}\right) = t$

$s = t$

Each triangle has altitude h: $\frac{h}{\sin(60)} = \frac{t}{\sin(90)} \to h = t\cdot\sin(60) = \frac{t\sqrt{3}}{2}$

So there are 6 of these triangles that make up the inscribed hexagon:

$$6\left[\frac{1}{2}\cdot\left(\frac{t\sqrt{3}}{2}\right)t\right] = 6\left[\frac{t^2\sqrt{3}}{4}\right] = \frac{3t^2\sqrt{3}}{2} = A_H$$

Now: What was the original question? #33, a couple pages back!

Show that the area of the inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

$$\sqrt{\left(\frac{3}{4}t^2\sqrt{3}\right)\cdot\left(3t^2\sqrt{3}\right)} = \sqrt{\frac{3^2t^4\cdot 3}{4}} = \frac{3\cdot t^2\sqrt{3}}{2}$$

Since $\dfrac{3t^2\sqrt{3}}{\frac{3t^2\sqrt{3}}{2}}\ \dfrac{[A_C]}{[A_H]} = 2$ and $\dfrac{\frac{3t^2\sqrt{3}}{2}}{\frac{3t^2\sqrt{3}}{4}}\ \dfrac{[A_H]}{[A_I]} = \dfrac{\frac{1}{2}}{\frac{1}{4}} = 2$

A_H is the geometric mean of A_C and A_I.

---

(34) Let a, b, and c denote real numbers that are not successive terms of an arithmetic progression or a geometric progression. If $a < b < c$, find a number which when added to a, b, and c yields consecutive terms of a geometric progression.

Solution: Let x be such a number. The geometric progression would be $a+x$, $b+x$, $c+x$.

What do we now: about the relationship between $a+x$, $b+x$, $c+x$, the consecutive terms of a geometric progression?

The common ratio!

$$\frac{b+x}{a+x} = \frac{c+x}{b+x}, \quad x \neq -a, \quad x \neq -b$$

Do the algebra! We want to solve for x.

$$(b+x)^2 = (c+x)(a+x) = ac + cx + ax + x^2$$
$$b^2 + 2bx + x^2 = x^2 + ac + cx + ax$$
$$b^2 - ac = cx + ax - 2bx = x(c+a-2b)$$
$$x = \frac{b^2-ac}{c+a-2b}$$ This is equivalent to $\frac{ac-b^2}{2b-a-c}$

---

(35) Discuss the validity of the theorem on p. 71 in case m is any nonnegative integer, $n=0$, $r \neq 0$, and $s \neq 0$.

1. $r^m r^n = r^{m+n}$
2. $(rs)^m = r^m s^m$
3. $(r^m)^n = r^{mn}$
4. $\frac{r^m}{r^n} = r^{m-n}$, if $m > n$
5. $\frac{r^m}{r^n} = \frac{1}{r^{m-n}}$, if $m < n$
6. $\left(\frac{s}{r}\right)^m = \frac{s^m}{r^m}$
7. If $r \notin \{-1, 0, 1\}$, $r^m = r^n$ if and only if $m = n$.

(p17 of Preliminaries Book 1)

If $n = m$ and $n \geq 1$, then $r^n = r^m$.
If $n = 0$ and $n = m$, then $m = 0$, and $b^n = 1$, $b^m = 1$, so $b^n = b^m$.

(36) Discuss the validity of the theorem on p. 85 in case $n = 0$. (p67 Preliminaries)

$$S_n = \frac{a - ar^n}{1 - r}$$

If $n = 0$ (and $r \neq 0$) then the sum is 0 since there are no terms.

3-5 THE BINOMIAL THEOREM

{1..8} Compute the indicated number.

(1) $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 5040$

(2) $\dfrac{10!}{8!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 10 \cdot 9 = 90$

(3) $\dfrac{6!}{3!\,2!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{(3 \cdot 2)(2)} = 6 \cdot 5 \cdot 2 = 60$

(4) $\dfrac{10!}{5!\,5!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2} = 2 \cdot 3 \cdot 2 \cdot 7 \cdot 3 = 252$

(5) $\dbinom{5}{3} = \dfrac{5 \cdot 4 \cdot 3}{3 \cdot 2} = \dfrac{5!}{3!\,2!} = \dfrac{5 \cdot 4}{2} = 10$

(6) $\binom{8}{5} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2} = 4 \cdot 7 \cdot 2 = \frac{8!}{5!\,3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2}$

$= 56$

(7) $\binom{6}{3} + \binom{6}{4} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} + \frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2}$

$= 2 \cdot 5 \cdot 2 + 5 \cdot 3 = 20 + 15 = 35$

$= \frac{6!}{3!\,3!} + \frac{6!}{4!\,2!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} + \frac{6 \cdot 5}{2}$

(8) $\binom{7}{4} + \binom{7}{5} = \binom{8}{5}$

$\frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5 \cdot 4 \cdot 3 \cdot 2} = 7 \cdot 5 + 7 \cdot 3 = 56$

$\frac{7!}{4!\,3!} + \frac{7!}{5!\,2!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} + \frac{7 \cdot 6}{2} = 7 \cdot 5 + 7 \cdot 3 = 56$

$\binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56$ ✓

(9) By trial determine if there is a positive integer n such that $n! = 50$.

No. $4! = 24$ and $5! = 120$

(10) Is there an integer $n > 1$ such that $n!$ is an odd number? Why or why not?

No since all $n!$ where $n > 1$, 2 is a factor.

{11..16} Express the given product or quotient by means of a single factorial expression. Assume $n > r > 0$.

(11) $(n+1)n! = (n+1)!$

(12) $\frac{(n+6)!}{n+6} = (n+5)!$

(13) $(n-r)(n-r-1)! = (n-r)!$

(14) $9 \cdot 8 \cdot 7 \cdot 6! = 9!$

(15) $n(n-1)(n-2)! = n!$

(16) $\frac{(n-r+1)!}{n-r+1} = (n-r)!$

(17) If the binomial expansion of $(a+b)^n$ has a middle term, what must be true of n?

n must be even.

(18) What is the sum of the exponents of x and y in the k^{th} term of the expansion of $(x+y)^n$? Assume $1 < k < n$.

The sum of exponents is always n.

(19) State the term in the expansion of $(a+b)^6$ having the coefficient $\binom{6}{2}$. The third term $\frac{6!}{2!4!} a^4b^2 = 15a^4b^2$

(20) State the term in the expansion of $(a+b)^7$ having the coefficient $\binom{7}{6}$. $n+1$ terms, so, of 8 terms, this is the 7^{th} term, $\binom{7}{6}ab^6$

{21, , 26} Simplify as far as possible, given x and y positive integers with $x > y$ and all denominators meaningful.

(21) $\dfrac{x!}{(x-1)!} = \dfrac{1}{x-1}$ (22) $\dfrac{(x+6)!}{(x+5)!} = x+6$

(23) $\dfrac{(x-y)!}{(x-y-1)!} = x+y$ (24) $\dfrac{x!}{(x-2)!} = x(x-1)$

(25) $\dfrac{(x+2)!}{x!} = (x+2)(x+1)$ (26) $\dfrac{(x-y+2)!}{(x-y+1)!} = x-y+2$

{27, , 32} Write the first 3 terms of each expansion and simplify.

(27) $(a+b)^9$: $a^9 + 9a^8b + 36a^7b^2$

(28) $(x-y)^7$: $x^7 + 7(x^6)(-y) + 21x^5(-y)^2$

$= x^7 - 7x^6y + 21x^5y^2$

(29) $(2a-3b)^5$: $(2a)^5 + 5(2a)^4(-3b) + 10(2a)^3(-3b)^2$

$= 32a^5 - 240a^4b + 720a^3b^2$

$= 16a^3(2a^2 - 15ab + 45b^2)$

tors

(30) $\left(x^2 - \frac{2}{x}\right)^4$: $(x^2)^4 + 4(x^2)^3\left(-\frac{2}{x}\right) + 6(x^2)^2\left(-\frac{2}{x}\right)^2$

$= x^8 - 8x^5 + 24x^2$

(31) $(x+3h)^4$: $x^4 + 4x^3(3h) + 6x^2(3h)^2$

$= x^4 + 12x^3h + 54x^2h^2$

(32) $\left(\frac{2}{a} + \frac{b}{2}\right)^5 = \left(\frac{2}{a}\right)^5 + 5\left(\frac{2}{a}\right)^4\left(\frac{b}{2}\right) + 10\left(\frac{2}{a}\right)^3\left(\frac{b}{2}\right)^2$

$= \frac{32}{a^5} + 40\frac{b}{a^4} + 20\frac{b^2}{a^3}$

{33 .. 38} Find all integral values of n for which the given statement is meaningful and true.

(33) $n! = 6 \rightarrow n = 3$

(34) $\frac{(n+1)!}{n!} = n+1 = 6 \longrightarrow n = 5$

(35) $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} = 10 \longrightarrow n^2 - n = 20$

$3!$

$n^2 - n - 20 = 0 \rightarrow (n-5)(n+4) = 0$

$n = -4$ or $n = 5$, but $n > 0$, so $n = 5$

(36) $6 < n! < 121$; $3! = 6$ $\therefore$ $n > 3$; $5! = 120$

$\therefore$ $n = 4$, $n = 5$

2

(37) $(n-1)! < 24$; $4! = 24$ $\therefore$ $(n-1) < 4 \rightarrow n < 5$

(38) $\frac{(n+1)!}{(n-1)!} = 12$; $n(n+1) = n^2 + n = 12 \rightarrow n^2 + n - 12 = 0$

$(n+4)(n-3) = 0$; $n = 3$

{39..44} Expand using the binomial theorem and express the result in simplest forms.

39 $(x+2)^4$ $\qquad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$$(x+2)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k} 2^k$$

$$= \binom{4}{0} x^4 2^0 + \binom{4}{1} x^3 \cdot 2^1 + \binom{4}{2} x^2 \cdot 2^2 + \binom{4}{3} x \cdot 2^3$$

$$+ \binom{4}{4} x^0 \cdot 2^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$$

40 $$(2a-b)^5 = \sum_{k=0}^{5} \binom{5}{k} (2a)^{5-k} (-b)^k$$

$$= \binom{5}{0} (2a)^5 (-b)^0 + \binom{5}{1} (2a)^4 (-b) + \binom{5}{2} (2a)^3 (-b)^2$$

$$+ \binom{5}{3} (2a)^2 (-b)^3 + \binom{5}{4} (2a)(-b)^4$$

$$+ \binom{5}{5} (2a)^0 (-b)^5$$

$$= 32a^5 + 5 \cdot 16a^4(-b) + 10 \cdot 8a^3b^2 + 10 \cdot 4a^2(-b)^3$$

$$+ 5 \cdot 2ab^4 + (-b)^5$$

$$= 32a^5 - 80a^4b + 80a^3b^2 - 40a^2b^3 + 10ab^4 - b^5$$

(41) $\left(x+\frac{2}{x}\right)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k}\left(\frac{2}{x}\right)^k$

$= \binom{4}{0} x^4 \left(\frac{2}{x}\right)^0 + \binom{4}{1} x^3 \left(\frac{2}{x}\right)^1 + \binom{4}{2} x^2 \left(\frac{2}{x}\right)^2$

$+ \binom{4}{3} x \cdot \left(\frac{2}{x}\right)^3 + \binom{4}{4} x^0 \left(\frac{2}{x}\right)^4$

$= x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}$

(42) $\left(x^2-1\right)^6 = \sum_{k=0}^{6} \binom{6}{k} \left(x^2\right)^{6-k} (-1)^k$

$= \binom{6}{0} (x^2)^6 (-1)^0 + \binom{6}{1} (x^2)^5 (-1)^1 + \binom{6}{2} (x^2)^4 (-1)^2$

$+ \binom{6}{3} (x^2)^3 (-1)^3 + \binom{6}{4} (x^2)^2 (-1)^4 + \binom{6}{5} (x^2)^1 (-1)^5$

$+ \binom{6}{6} (x^2)^0 (-1)^6 = x^{12} - 6x^{10} + 15x^8 - 20x^6$

$+ 15x^4 - 6x^2 + 1$

(43) $\left(\frac{2}{x}+\frac{3}{y}\right)^3 = \sum_{k=0}^{3} \binom{3}{k} \left(\frac{2}{x}\right)^{3-k} \left(\frac{3}{y}\right)^k$

$= \binom{3}{0} \left(\frac{2}{x}\right)^3 \left(\frac{3}{y}\right)^0 + \binom{3}{1} \left(\frac{2}{x}\right)^2 \left(\frac{3}{y}\right)^1 + \binom{3}{2} \left(\frac{2}{x}\right)^1 \left(\frac{3}{y}\right)^2$

$+ \binom{3}{3} \left(\frac{2}{x}\right)^0 \left(\frac{3}{y}\right)^3 = \frac{8}{x^3} + \frac{36}{x^2 y} + \frac{54}{x y^2} + \frac{27}{y^3}$

(44) $(2x-3h)^3 = \sum_{k=0}^{3} \binom{3}{k}(2x)^{3-k}(-3h)^k$

$= \binom{3}{0}(2x)^3(-3h)^0 + \binom{3}{1}(2x)^2(-3h)^1$

$+ \binom{3}{2}(2x)^1(-3h)^2 + \binom{3}{3}(2x)^0(-3h)^3$

$= 8x^3 - 36x^2h + 54xh^2 - 27h^3$

{45..50} Write the indicated term in the expansion of the binomial.

(45) $(a+b)^{10}$; 5^{th}

Note: the r^{th} term in the expansion of

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

is $\binom{n}{r-1} a^{n-(r-1)} b^{r-1}$

So, the 5^{th} term in the expansion of $(a+b)^{10}$

is $\binom{10}{4} a^{10-(5-1)} b^{5-1} = \frac{10!}{4!6!} a^{10-4} b^4$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} a^6 b^4 = 210a^6b^4$

(46) $(x-y)^7$; 4th term is $\binom{7}{4-1} x^{7-(4-1)}(-y)^{4-1}$

$$= \frac{7!}{3!\,4!} x^{7-3}(-y)^3 = \frac{7\cdot6\cdot5}{3\cdot2} x^4(-y)^3$$

$$= -35x^4y^3$$

(47) $\left(\frac{2}{x}+\frac{x}{2}\right)^8$; middle term

When $n=8$, there are $n+1=9$ terms

So, the middle term is $r=5$.

$$\binom{8}{5-1}\left(\frac{2}{x}\right)^{8-(5-1)}\left(\frac{x}{2}\right)^{5-1} = \frac{8!}{4!\,4!}\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right)^4$$

$$= \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}\left(\frac{16}{x^4}\right)\left(\frac{x^4}{16}\right) = 7\cdot2\cdot5 = 70$$

(48) $(3x^2-2y)^6$; middle term

Since $n=6$ there are $n+1=7$ terms

The middle term is $r=4$

$$\binom{6}{4-1}(3x^2)^{6-(4-1)}(-2y)^{4-1} = \frac{6!}{3!\,3!}(3x^2)^3(-2y)^3$$

$$= \frac{6\cdot5\cdot4}{3\cdot2}(27x^6)(-8y^3) = -(20\cdot27\cdot8)x^6y^3$$

$$= -4320x^6y^3$$

(49) $(a+b)^n$; $(n-2)^{nd}$ term

$$\binom{n}{(n-2)-1} a^{n-(n-3)} b^{n-3} = \frac{n!}{(n-3)!\,(n-(n-3))!} a^3 b^{n-3}$$

$$= \frac{n!}{(n-3)!\,3!} a^3 b^{n-3} = \frac{n(n-1)(n-2)}{6} a^3 b^{n-3}, \quad n>2$$

(50) $(x+y)^n$; $(n-r)^{th}$ term

$$\binom{n}{(n-r)-1}(x)^{n-[(n-r)-1]}(y)^{(n-r)-1}$$

$$= \binom{n}{n-r-1} x^{r+1} y^{n-r-1}$$

$$= \frac{n!}{(n-r-1)!} x^{r+1} y^{n-r-1}$$

$$= \frac{n(n-1)\cdots(n-r)}{(n-[n-r-1])!} x^{r+1} y^{n-r-1}$$

$$= \frac{n(n-1)\cdots(n-r)}{r+1} x^{r+1} y^{n-r-1}$$

(51) Write the term in the expansion of $(3x-2y)^7$ in which the exponent of y is 4.

$$\binom{7}{4}(3x)^3(-2y)^4 = \frac{7!}{4!\,3!} 27x^3(16y^4)$$

$$= \frac{7\cdot 6\cdot 5}{3\cdot 2} \cdot 432\, x^3 y^4 = 15120\, x^3 y^4$$

(52) Write the term in the expansion of $(a^2+2b)^{14}$ in which the exponent of "a" is 8.

Solution: The exponent of "a" is 8 when the exponent of (a^2) is 4. $4 = 14-k$ when $k=10$, so the 11^{th} term, i.e., $r=11$

$$\binom{14}{11-1}(a^2)^{14-(11-1)}(2b)^{11-1} = \frac{14!}{10!\,4!}(a^2)^4(2b)^{10}$$

$$= \frac{14\cdot 13\cdot 12\cdot 11}{4\cdot 3\cdot 2}a^8\cdot(1024\, b^{10}) = 7\cdot 13\cdot 11\cdot 1024\, a^8 b^{10}$$

$$= 1025024\, a^8 b^{10}$$

(53) If the coefficients of the 4^{th} and 16^{th} terms in the expansion of $(r+s)^n$ are equal, find the middle term.

The fourth term is $\binom{n}{4-1} r^{n-(4-1)} s^{4-1} = \frac{n!}{3!\,(n-3)!} r^{n-3} s^3$

$= \frac{n(n-1)(n-2)}{6}\cdot r^{n-3} s^3$ has the same coeffecient as

$\frac{n!}{(n-3)!\,(n-[n-3])} r^3 s^{n-3}$ There are 11 terms between the fourth and sixteenth terms, thus the middle term is the 6th term between them, or the 10^{th} term in the expansion.

Thus there are 19 terms. $n=18$ and $k=9$ for the 10^{th} term.

$$\binom{18}{9} r^{18-9} s^9 = \frac{18!}{9!\,9!} r^9 s^9 =$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} r^9 s^9$$

$$= nCr(18,9) \cdot r^9 s^9 = 48620\, r^9 s^9$$

54 If the coeffecients of the 6^{th} and 16^{th} terms in the expansion of $(a-b)^n$ are equal, find the third term.

Solution: There are 9 terms between the 6^{th} and 16^{th} terms, so the 5^{th} between them is the middle term, or the 11^{th} term in the expansion.

There are thus 21 terms so $n = 20$

To find the 3^{rd} term, $r = 3$; that is $k = 2$

$$\binom{20}{2} a^{18} b^2 = \frac{20!}{2!\,18!} a^{18} b^2 = \frac{20 \cdot 19}{2} a^{18} b^2$$

$$= 190\, a^{18} b^2$$

55

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56

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{55, 56} Verify the given statement by direct computation.

(55) $\sum_{k=0}^{3} \binom{3}{k} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$

$= 1 + 3 + 3 + 1 = 8 = 2^3$ ✓

(56) $\sum_{k=0}^{4} \binom{4}{k} = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$

$= 1 + 4 + 6 + 4 + 1 = 16 = 2^4$ ✓

{57..60} Find the value to the nearest thousandth.

(57) $(1 + 0.02)^5$

Solution: Consider $k < 3$ for value to nearest thousandth. For $k \geq 3$ the term would be less than or equal to $(10)(.000008)$

$$\sum_{k=0}^{2} \binom{5}{k} 1^{5-k} (0.02)^k = \binom{5}{0} 1^5 (0.02)^0 + \binom{5}{1} 1^4 (0.02)$$

$$+ \binom{5}{2} 1^3 (0.02)^2 = 1 + 5(0.02) + 10(0.0004)$$

$= 1 + 0.1 + 0.004 = 1.104$

(58) $(1 - 0.01)^4$

$$\sum_{k=0}^{2} \binom{4}{k} 1^{4-k} (-0.01)^k = \binom{4}{0} 1^4 (-0.01)^0 + \binom{4}{1} 1^3 (-0.01)$$

$$+ \binom{4}{2} 1^2 (-0.01)^2 = 1 - 0.04 + 0.0006$$

$= 0.961$

(59) $(1.03)^5 = (1+0.03)^5$

$$\sum_{k=0}^{2} \binom{5}{k} 1^{5-k} \cdot (0.03)^k = \binom{5}{0} 1^5 \cdot (0.03)^0$$

$$+ \binom{5}{1} 1^4 \cdot (0.03)^1 + \binom{5}{2} 1^3 \cdot (0.03)^2$$

$$= 1 + 5(0.03) + 10(0.0009) = 1.1509$$

$$\approx 1.151$$

(60) $(0.98)^6 = (1-0.02)^6$

$$\sum_{k=0}^{2} \binom{6}{k} 1^{6-k} (-0.02)^k$$

$$= \binom{6}{0} 1^6 + \binom{6}{1} 1^5 (-0.02)^1 + \binom{6}{2} 1^4 (-0.02)^2$$

$$= 1 - 0.12 + 0.006 = 0.886$$

Prove each statement. Assume $n \in \mathbb{N}$, $k \in \mathbb{N}$, and $k \leq n$.

(61) $\binom{n}{k} = \binom{n}{n-k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}$$

(62

(63

k

(62) $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}\cdot\frac{(n-k+1)}{(n-k+1)}$$

$$= \frac{k\cdot(n!)}{k\cdot(k-1)!\,(n-k+1)!} + \frac{n!\,(n-k+1)}{k!\,(n-k+1)!}$$

$$= \frac{k(n!)}{k!\,(n-k+1)!} + \frac{n!\,(n-k+1)}{k!\,(n-k+1)!} = \frac{k(n!) + n!\,(n-k+1)}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(k+n-k+1)}{k!\,(n-k+1)!} = \frac{n!\,(n+1)}{k!\,(n-k+1)!} = \frac{(n+1)!}{k!\,(n-k+1)!}$$

$$= \binom{n+1}{k} = \frac{(n+1)!}{k!\,(n+1-k)!}$$

(63) If $(a+b)^n + (a-b)^n$ yields, when expanded and simplified, an expression containing 4 terms, what conclusion can you reach about n? about a and b?

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-b)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \left[b^k + (-b)^k\right]$$

$$(a+b)^n + (a-b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\left[b^k + (-b)^k\right]$$

if k is odd, $(-b)^k < 0$ and $[b^k + (-b)^k] = 0$
if k is even, $(-b)^k > 0$ and $[b^k + (-b)^k] = 2b^k$

For the expansion to have 4 terms, there must be exactly 4 even $k = 0, 2, 4, 6$;
so $n = 6$ or $n = 7$.

$$\sum_{k=0}^{6} a^{6-k}\left[b^k + (-b)^k\right] = \binom{6}{0} a^6\left[b^0 + (-b)^0\right]$$
$$+ \binom{6}{2} a^4\left[b^2 + (-b)^2\right] + \binom{6}{4} a^2\left[b^4 + (-b)^4\right]$$
$$+ \binom{6}{6} a^0\left[b^6 + (-b)^6\right]$$

$$= 2a^6 + 15a^4b^2 + 15a^2b^4 + 2b^6$$

For $n=7$, the 8^{th} term is 0 and the expansion still has 4 terms

$a \neq 0$ and $b \neq 0$, otherwise expansion would have one or no terms.

$a \neq b$ otherwise there would only be one term, ca^6 or cb^6.

(64) Expand $(a+b-z)^4$ through repeated use of the binomial theorem.

$$(a+b-z)^4 = [a+(b-z)]^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k}(b-z)^k$$

$$= \binom{4}{0} a^4 (b-z)^0 + \binom{4}{1} a^3 (b-z) + \binom{4}{2} a^2 (b-z)^2$$

$$+ \binom{4}{3} a (b-z)^3 + \binom{4}{4} a^0 (b-z)^4$$

$$= a^4 + 4a^3 \cdot (b-z) + 6a^2 (b-z)^2 + 4a(b-z)^3$$

$$+ (b-z)^4 = a^4 + 4a^3 b - 4a^3 z + 6a^2 \left(\sum_{k=0}^{2} \binom{2}{k} b^{2-k} (-z)^k\right)$$

$$+ 4a\left[\sum_{k=0}^{3} \binom{3}{k} b^{3-k} (-z)^k\right] + \sum_{k=0}^{4} \binom{4}{k} b^{4-k} (-z)^k$$

$$= a^4 + 4a^3 b - 4a^3 z + 6a^2 \left(b^2 + 2b(-z) + z^2\right)$$

$$+ 4a\left[b^3 + 3b^2(-z) + 3b(-z)^2 + (-z)^3\right]$$

$$+ \left[b^4 + 4b^3(-z) + 6b^2(-z)^2 + 4b(-z)^3 + z^4\right]$$

$$= a^4 + 4a^3 b - 4a^3 z + 6a^2 b^2 - 12a^2 \cdot b \cdot z + 6a^2 z^2$$

$$+ 4ab^3 - 12ab^2 z + 12a \cdot b \cdot z^2 - 4az^3 + b^4 - 4b^3 z$$

$$+ 6b^2 z^2 - 4bz^3 + z^4$$

(65) By the method of mathematical induction prove that in the expansion of $(a+b)^n$, $a>0$ and $b>0$, the sum of the coefficients is 2^n.

Let S be the set of integers $n>0$ for which the statement is true.

(1) $1 \in S$: $(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k$

$= \binom{1}{0} a + \binom{1}{1} b = a+b$

Sum of coefficients $1+1=2^1$

(2) Assume $x \in S$ and the sum of the coefficients of the expansion $(a+b)^x$ is 2^x.

Then $x+1 \in S$ and the sum of the coefficients of the expansion of $(a+b)^{x+1}$ is 2^{x+1}:

$$(a+b)^{x+1} = (a+b)^1 (a+b)^x = a(a+b)^x + b(a+b)^x$$

Multiplication of $(a+b)^x$ by a or b does not change the sum of the coefficients, only the exponent of a or b.

The sum of the coefficients of $a(a+b)^x$

$= $ the sum of the coefficents of $(a+b)^x$

$= $ the sum of the coefficents of $b(a+b)^x$.

So the sum of the coefficients of $(a+b)^{x+1}$ is $2^x + 2^x = 2(2^x) = 2^{x+1}$

(66) Using the method of mathematical induction, prove that in the expansion of $(a-b)^n$, $a > 0$ and $b > 0$, the sum of the coefficients is zero.

Let S be the set of integers $n > 0$ for which the statement is true:

(1) $1 \in S$: $(a-b)^1 = 1 \cdot a - 1 \cdot b$, $1 - 1 = 0$

(2) Assume $x \in S$ and the sum of the coefficients of the expansion $(a-b)^x$ is 0.
Then $x+1 \in S$ and the sum of the coefficents of the expansion of $(a-b)^{x+1}$ is 0:

$$(a-b)^{x+1} = (a-b)(a-b)^x = a(a-b)^x - b(a-b)^x.$$

By reasoning similar to exercise 65 and from assumption the sum of coefficents is $0 - 0 = 0$.

INFINITE SEQUENCES AND SERIES

3-6 LIMIT OF A SEQUENCE

{1,,6} a_n represents the general term of a sequence. For each sequence tell whether, as n increases, the absolute value of the difference between successive terms increases, decreases, or remains constant.

① $a_n = \frac{1}{n}$: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

$\left|1-\frac{1}{2}\right| = \frac{1}{2}$, $\left|\frac{1}{2}-\frac{1}{3}\right| = \frac{1}{6}$, $\left|\frac{1}{3}-\frac{1}{4}\right| = \frac{1}{12}$

decreases (by brute force)

Is there a more mathematically "sophisticated" method to deduce this conclusion?

$$|a_n - a_{n+1}| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{(n+1)-n}{n(n+1)}\right| = \frac{1}{n(n+1)};$$

$\therefore$ decreases

② $a_n = 2n-3$: $|a_n - a_{n+1}| = |(2n-3)-(2(n+1)-3)|$

$= |(2n-3)-(2n-1)| = |-2| = 2$;

remains constant

③ $a_n = (-1)^n \cdot (2n)$;

$$|a_n - a_{n+1}| = \left| (-1)^n(2n) - (-1)^{n+1}(2n+2) \right|$$

$$= \left| (-1)^n(2 \cdot n) - (-1)(-1)^n(2n+2) \right|$$

$$= \left| (-1)^n \left[2n - \left((-1)(2n+2) \right) \right] \right| = \left| (-1)^n(2n+2n+2) \right|$$

$$= \left| (-1)^n(4n+2) \right| = 4n+2 \text{ ; increases}$$

④ $a_n = \dfrac{2n+3}{4n+1}$; $|a_n - a_{n+1}| = \left| \left(\dfrac{2n+3}{4n+1}\right) - \left(\dfrac{2n+5}{4n+5}\right) \right|$

$$= \left| \frac{(4n+5)(2n+3) - (2n+5)(4n+1)}{(4n+1)(4n+5)} \right|$$

$$= \left| \frac{(8n^2+22n+15) - (8n^2+22n+5)}{(4n+1)(4n+5)} \right| = \left| \frac{-10}{16n^2+24n+5} \right|$$

$$= \frac{10}{(4n+1)(4n+5)} \text{ ; decreases}$$

⑤ $a_n = \dfrac{n^2+3}{n+2}$; $|a_n - a_{n+1}| = \left| \left(\dfrac{n^2+3}{n+2}\right) - \left(\dfrac{(n+1)^2+3}{n+3}\right) \right|$

$$= \left| \left(\frac{n^2+3}{n+2}\right) - \left(\frac{n^2+2n+1+3}{n+3}\right) \right| = \left| \frac{(n^2+3)(n+3) - (n^2+2n+4)(n+2)}{(n+2)(n+3)} \right|$$

$$= \left| \frac{(n^3+3n^2+3n+9)-(n^3+2n^2+4n+2n^2+4n+8)}{(n+2)(n+3)} \right|$$

$$= \left| \frac{-n^2-5n+1}{(n+2)(n+3)} \right| = \left| \frac{-(n^2+5n-1)}{(n+2)(n+3)} \right|;$$

$$= \left| \frac{-(n^2+5n-1)}{n^2+5n+6} \right| = \left| \frac{-n^2-5n+1}{n^2+5n+6} \right|$$

$$\begin{array}{r|l} & -1 \\ \hline n^2+5n+6 & -n^2-5n+1 \\ & -n^2-5n-6 \\ \hline & \quad\quad 7 \end{array}$$

$$= \left| -1 + \frac{7}{n^2+5n+6} \right|$$

$$= 1 - \frac{7}{n^2+5n+6}$$

$\therefore$ increases

(6) $a_n = \dfrac{(n+1)!}{n!}$; $|a_n - a_{n+1}|$

$$= \left| \left(\frac{(n+1)!}{n!}\right) - \left(\frac{(n+2)!}{(n+1)!}\right) \right|$$

$$= \left| \frac{(n+1)(n+1)! - (n+2)!}{(n+1)!} \right| = \left| \frac{(n+1)!\,((n+1)-(n+2))}{(n+1)!} \right|$$

$$= \left| n+1-n-2 \right| = |-1| = 1 \text{; remains constant}$$

{7.12} Classify each of the following statements as true or false.

⑦ A term of a convergent sequence can be equal to the limit of the sequence.
TRUE. Example: $1, 1, 1, 1, \ldots, 1, \ldots$

⑧ If the limit of a convergent sequence is a positive number, all terms of the sequence are positive numbers.

FALSE. Example: $-\frac{1}{2}, 0, \frac{1}{6}, \frac{1}{4}, \ldots, \frac{1}{2}-\frac{1}{n}, \ldots,$ with a limit of $\frac{1}{2}$.

⑨ If the terms of a sequence are alternately positive and negative numbers, the sequence cannot be convergent.

FALSE. Example: $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^n\left(\frac{1}{n}\right), \ldots$ with limit 0.

⑩ If a_n is the tenth term of a convergent sequence with limit L, then for every two successive terms a_k and a_{k+1}, $|a_k - L| \geq |a_{k+1} - L|$.

FALSE. Example: $1, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{3}{5}, \ldots, \frac{3}{n} \ldots$

$L = 0, \; n = 2, \; a_K = \frac{1}{3}, \; a_{K+1} = \frac{3}{4}$

$\frac{1}{3} < \frac{3}{4}$ so $\left|\frac{1}{3} - 0\right| \not> \left|\frac{3}{4} - 0\right|$

(11) If $\lim_{n\to\infty} a_n = T$ and $\lim_{n\to\infty} b_n = T$ then a_n and b_n designate the same sequence.

FALSE. Example: $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$
and $1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots$ both have $T=0$.

(12) If $\lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \left(3 + \frac{1}{n}\right) = 3$

True. $\lim_{n\to\infty} (a_n + b_n) = A + B$

{13.. 18} Find the indicated limit, given $\lim_{n\to\infty} a_n = k^2 - 1$, $\lim_{n\to\infty} b_n = k+1$, and $|k| \neq 1$.

(13) $\lim_{n\to\infty} 3a_n = 3k^2 - 3$

(14) $\lim_{n\to\infty} (a_n + b_n) = (k^2 - 1) + (k+1) = k^2 + k$

15. $$\lim_{n\to\infty}(a_n - b_n) = (k^2-1)-(k+1)$$
$$= k^2-k-2$$

16. $$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{k^2-1}{k+1} = \frac{(k+1)(k-1)}{k+1} = k-1$$
$$k \neq -1$$

17. $$\lim_{n\to\infty}\left(\frac{a_n+b_n}{b_n}\right) = \frac{(k^2-1)+(k+1)}{k+1} = \frac{(k+1)(k-1)}{k+1} + \frac{k+1}{k+1}$$

$$= k-1+1 = k$$

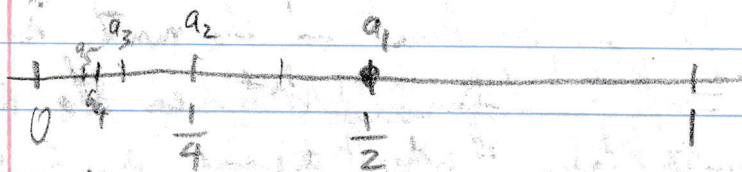
18. $$\lim_{n\to\infty}\left(\frac{1}{a_n}-\frac{1}{b_n}\right) = \left(\frac{1}{k^2-1}-\frac{1}{k+1}\right)$$

$$= \frac{(k+1)-(k^2-1)}{(k^2-1)(k+1)} = \frac{-k^2+k+2}{(k^2-1)(k+1)} = \frac{-(k^2-k-2)}{(k^2-1)(k+1)}$$

$$= \frac{-(k-2)(k+1)}{(k^2-1)(k+1)} = \frac{-k+2}{k^2-1}$$

{19., 24} Determine whether the given sequence is convergent or divergent by means of a number-line representation. For each convergent sequence find the limit and compute the difference between the limit and the ninth and tenth terms.

(19) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n}, \ldots$

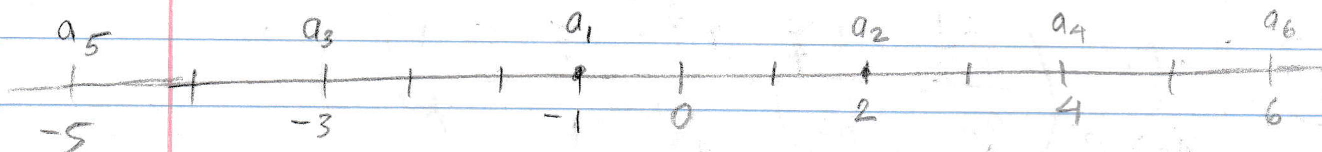


$\lim_{n \to \infty} \frac{1}{2n} = 0 \qquad \left|\frac{1}{2(9)}\right| > \left|\frac{1}{2(10)}\right|$

$\frac{1}{18} > \frac{1}{20}$

---

(20) $-1, 2, -3, 4, \ldots, (-1)^n n, \ldots$



divergent

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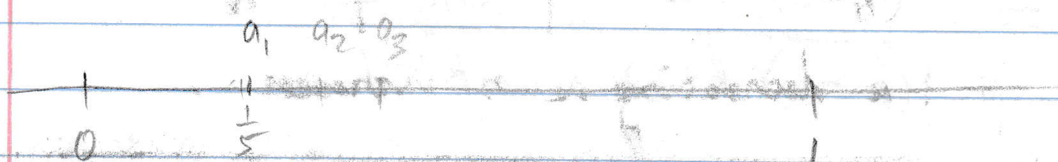
(21) $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \ldots, \frac{n}{2n+3}, \ldots$



$$\lim_{n \to \infty} \frac{n}{2n+3} = \lim_{n \to \infty} \frac{1}{2 + \frac{3}{n}} = \frac{1}{2}$$

$$\left|\frac{1}{2} - \frac{9}{2(9)+3}\right| = \left|\frac{1}{2} - \frac{9}{21}\right| = \left|\frac{1}{2} - \frac{3}{7}\right| = \frac{1}{14}$$

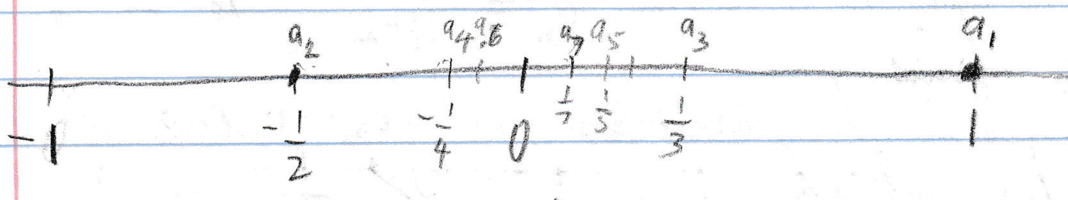
$$\left|\frac{1}{2} - \frac{10}{23}\right| = \left|\frac{23}{46} - \frac{20}{46}\right| = \frac{3}{46}$$

$\frac{1}{14} > \frac{3}{46}$ means $|a_9 - L| > |a_{10} - L|$

The difference between the limit and the ninth term is greater than the difference between the limit and the tenth term, which means $a_9 < a_{10}$.
The means a_n is increasing as n increases.

---

(22) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1} \cdot \frac{1}{n}, \ldots$



a_6
6

Evidently, a_n is approaching 0.

$$\lim_{n \to \infty} (-1)^{n+1} \frac{1}{n} = 0$$

$$a_9 = (-1)^{10} \frac{1}{9} = \frac{1}{9} ; \quad |a_9 - 0| = \frac{1}{9}$$

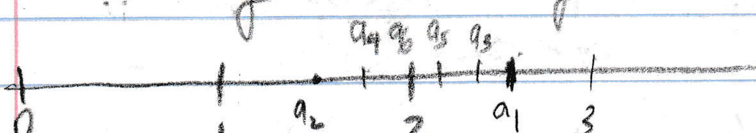
$$a_{10} = \frac{(-1)^{11}}{10} = -\frac{1}{10} ; \quad \left|-\frac{1}{10} - 0\right| = \frac{1}{10} \qquad a_{10} < a_9$$

$|a_n|$ is decreasing as n increases.

---

(23) $\left(2 + \frac{1}{2}\right), \left(2 - \frac{1}{2}\right), \left(2 + \frac{1}{3}\right), \left(2 - \frac{1}{3}\right), \ldots$

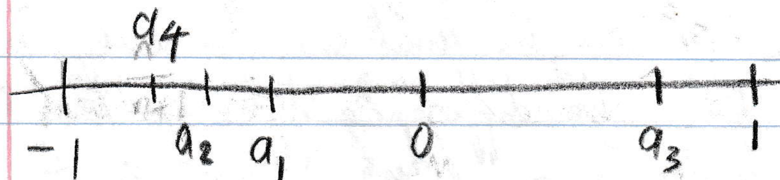
What is a_n? It alternates from $2 + \frac{1}{n+1}$ to $2 - \frac{1}{n+1}$ but n only increases every other term. What's wrong here?



converges to 2

$$|a_9 - 2| = \left|2 + \frac{1}{6} - 2\right| = \frac{1}{6} ; \quad |a_{10} - 2| = \left|2 - \frac{1}{6} - 2\right| = \frac{1}{6}$$

(24) $-\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$



divergent

(25) Given that the limit L of the sequence with n^{th} term $a_n = \frac{n}{n+2}$ is 1, find the smallest integral value of n for which $|L - a_n| < \frac{1}{7}$.

$$\left|1 - \frac{n}{n+2}\right| < \frac{1}{7} \rightarrow \left|\frac{n+2-n}{n+2}\right| < \frac{1}{7}$$

$$\frac{2}{n+2} < \frac{1}{7} \rightarrow \frac{n+2}{2} > 7$$

$$n+2 > 14 \rightarrow n > 12$$

$\therefore$ the smallest integral value of n for which $\left|1 - \frac{n}{n+2}\right| < \frac{1}{7}$ is $n = 13$.

(26) Given that the limit L of the sequence with n^{th} term $a_n = \dfrac{n^2}{2n^2+1}$ is $\dfrac{1}{2}$, find the smallest integral value of n for which $|L - a_n| < \dfrac{1}{66}$.

$$\left| \frac{1}{2} - \frac{n^2}{2n^2+1} \right| < \frac{1}{66} \rightarrow \left| \frac{(2n^2+1) - 2n^2}{2(2n^2+1)} \right| < \frac{1}{66}$$

$$\frac{1}{4n^2+2} < \frac{1}{66} \rightarrow 4n^2+2 > 66 \rightarrow 4n^2 > 64$$

$$n^2 > 16 \rightarrow n > 4 \text{ or } n < -4 \text{ but } n > 0$$

so the smallest integral value $n = 5$

(27) Assuming that the limit L of the sequence with n^{th} term $a_n = \dfrac{3n}{2n^2+7}$ is 0, find the smallest value of n for which $|L - a_n| < \dfrac{1}{5}$.

$$\left| \frac{-3n}{2n^2+7} \right| < \frac{1}{5} \rightarrow 2n^2+7 > (3n) \cdot 5 = 15n$$

$$2n^2 - 15n + 7 > 0 \rightarrow (2n-1)(n-7) > 0$$

Either $2n-1 > 0$ and $n-7 > 0$ or $2n-1 < 0$ and $n-7 < 0$

$2n > 1$ and $n > 7$ OR $2n < 1$ and $n < 7$

$n > \frac{1}{2}$ $\therefore n = 8$ $\qquad$ $n < \frac{1}{2}$ $\therefore n = 0$

but 0 is not in $\mathbb{N}$ so $n = 8$

(28) Prove that the sequence with n^{th} term

$$a_n = \frac{n^2+1}{n+6} \text{ diverges.}$$

$$\left[\text{HINT: } \frac{n^2+1}{n+6} = \frac{(n^2-36)+37}{n+6} = n-6+\frac{37}{n+6}\right]$$

$$\begin{array}{r|l} & n-6 \\ \hline n+6 & n^2+0n+1 \\ & n^2+6n \\ \hline & -6n \\ & -6n-36 \\ \hline & 37 \end{array} \quad \left. \begin{array}{l} a_n = n-6+\dfrac{37}{n+6} \\ \lim\limits_{n\to\infty} a_n = \infty \end{array} \right.$$

$\therefore$ the sequence diverges.

{29..36} Find $\lim\limits_{n\to\infty} a_n$ for each sequence that is convergent. [Hint: Express a_n in a form which permits you to make use of the fact that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.]

(29) $a_n = \dfrac{n}{n+3}$

$$\lim_{n\to\infty}\left[\frac{\frac{n}{n}}{\frac{n}{n}+\frac{3}{n}}\right] = \lim_{n\to\infty}\left[\frac{1}{1+\frac{3}{n}}\right] = 1$$

(30)

(31)

(32)

(33)

(34)

(35)

(36)

{37.

the

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(30) $a_n = \dfrac{n^2}{n+1}$; $\lim_{n\to\infty}\left[\dfrac{\frac{n^2}{n^2}}{\frac{n}{n^2}+\frac{1}{n^2}}\right]$

$= \lim_{n\to\infty} \dfrac{1}{\frac{1}{n}+\frac{1}{n^2}} = \infty$ $\therefore$ divergent.

(31) $a_n = \dfrac{3n-2}{n+4}$; $\lim_{n\to\infty}\left[\dfrac{3-\frac{2}{n}}{1+\frac{4}{n}}\right] = 3$

(32) $a_n = \dfrac{n!}{n} = (n-1)!$

$\lim_{n\to\infty}\left[(n-1)!\right] = \infty$ $\therefore$ divergent.

(33) $a_n = \dfrac{n^2+1}{3n^2-n}$; $\lim_{n\to\infty}\left[\dfrac{1+\frac{1}{n^2}}{3-\frac{1}{n}}\right] = \dfrac{1}{3}$

(34) $a_n = \dfrac{2n^2+3n-1}{5n^2+4n+2}$; $\lim_{n\to\infty}\left[\dfrac{2+\frac{3}{n}-\frac{1}{n^2}}{5+\frac{4}{n}+\frac{2}{n^2}}\right] = \dfrac{2}{5}$

(35) $a_n = \left(1+\frac{2}{n}\right)\left(2-\frac{1}{n}\right)$; $\lim_{n\to\infty} a_n = (1)(2) = 2$

(36) $a_n = \left(\dfrac{4n+1}{n}\right)^{10}$; $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\dfrac{4+\frac{1}{n}}{1}\right)^{10} = 4^{10}$

{37..44} (a) State at least one appropriate n^{th} term of the suggested sequence. (b) if the resulting sequence is convergent, find the limit by using the method suggested for ex. 29-36.

(37) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ $\left(\ldots \frac{1}{9}, \frac{1}{11}\right.$ $\underset{n=5}{\uparrow}$ $\underset{n=6}{\uparrow}$

(a) $a_n = \frac{1}{2n-1}$ $\therefore$ $a_5 = \frac{1}{2(5)-1} = \frac{1}{9}$

(b) $\lim_{n\to\infty} a_n = 0$

(38) $1, 3!, 5!, 7!, \ldots$

(a) $a_n = (2n-1)!$ $\therefore$ $a_5 = (2(5)-1)! = 9!$

(b) $\lim_{n\to\infty} (2n-1)! = \infty$ $\therefore$ divergent

(39) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

(a) $a_n = \frac{n+1}{n}$ $\therefore$ $a_5 = \frac{6}{5}$

(b) $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$

(40) $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots$

(a) $a_n = \frac{n}{2n-1}$ $\therefore$ $a_5 = \frac{5}{9}$

(b) $\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$

(41) $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$

(a) $a_n = \frac{n^2}{n+1}$ $\therefore$ $a_5 = \frac{25}{6}$

(b) $\lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \infty$ $\therefore$ divergent

(42) $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \dots$

(a) $a_n = (-1)^n \frac{n}{n+1} \quad \therefore \quad a_5 = (-1)^5 \frac{5}{6} = -\frac{5}{6}$

(b) Divergent since alternating values approach 1 and -1 ; $1-1+1-1+1\dots$ is divergent.

(43) $\frac{3}{1}, \frac{4}{3}, \frac{5}{5}, \frac{6}{7}, \dots$

(a) $a_n = \frac{n+2}{2n-1} \quad \therefore \quad a_5 = \frac{7}{9}, \quad a_{10} = \frac{12}{19}$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{2-\frac{1}{n}} = \frac{1}{2}$

(44) $-\frac{1}{2}, -\frac{2}{5}, -\frac{3}{10}, -\frac{4}{17}, \dots$

(a) $a_n = \frac{-n}{n^2+1} \quad \therefore \quad a_5 = \frac{-5}{26}, \quad a_6 = \frac{-6}{37}$

This appears to be approaching 0 "from the left"

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-\frac{1}{n}}{1+\frac{1}{n^2}} = 0$

(45) Given the sequence whose x^{th} term is

$$a_x = \frac{s_0 x^m + s_1 x^{m-1} + s_2 x^{m-2} + \dots + s_{m-1} x + s_m}{t_0 x^n + t_1 x^{n-1} + t_2 x^{n-2} + \dots + t_{n-1} x + t_n} \quad (t_0 \neq 0)$$

Show that (a) if $m < n$, $\lim_{x\to\infty} a_x = 0$. (b) if $m = n$, $\lim_{x\to\infty} a_x = \frac{s_0}{t_0}$

(a) Show that if $m < n$, $\lim_{x\to\infty} a_x = 0$

if $m < n$, then $\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} \frac{1}{x^{n-m}} = 0$

so $\lim_{x\to\infty} \frac{s_0}{x^{n-m}} + \frac{s_1}{x^{n-m-1}} + \dots + \frac{s_{m-1}}{x^{n-1}} + \frac{s_m}{x^n}$

$$= \frac{0+0+\dots+0}{t_0+0+\dots+0} = 0$$

(b) Show that if $m = n$, $\lim_{x\to\infty} a_x = \frac{s_0}{t_0}$

$$\lim_{x\to\infty} \frac{s_0 + \frac{s_1}{x^1} + \frac{s_2}{x^2} + \dots + \frac{s_{m-1}}{x^{n-1}} + \frac{s_m}{x^n}}{t_0 + \frac{t_1}{x^1} + \frac{t_2}{x^2} + \dots + \frac{t_{n-1}}{x^{n-1}} + \frac{t_n}{x^n}}$$

$$= \frac{s_0 + 0 + 0 + \dots + 0}{t_0 + 0 + 0 + \dots + 0} = \frac{s_0}{t_0}$$

---

(46) Prove that if $0 < r < 1$ then $\lim_{n\to\infty} r^n = 0$

[Hint: Substitute $\frac{1}{1+k}$ $(k > 0)$ for r and apply THEOREM (if $a \geq -1$, then $(1+a)^n \geq 1+na$) to $(1+k)^n$]

Letting $r = \frac{1}{1+k}$ $(k > 0)$, if $0 < \frac{1}{1+k} < 1$,

prove $\lim\limits_{n\to\infty} \left(\frac{1}{1+k}\right)^n = 0$.

$$0 < \left(\frac{1}{1+k}\right)^n = \frac{1}{(1+k)^n} \leq \frac{1}{1+n\cdot k},$$

since $(1+k)^n \geq 1 + n\cdot k$ (by theorem p. 72 TEXT)

$$\lim_{n\to\infty} \frac{1}{(1+k)^n} \leq \lim_{n\to\infty} \frac{1}{1+n\cdot k}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}+k} = \frac{0}{k} = 0$$

Since $\left(\frac{1}{1+k}\right)^n$ is never negative,

then $\lim\limits_{n\to\infty} \frac{1}{(1+k)^n} = 0$.

---

(47) Given $a_n = \frac{1-n}{1+n}$, Prove $\lim\limits_{n\to\infty} a_n = -1$ by showing that for any positive integer n greater than $\frac{2}{h} - 1$, $|a_n - (-1)| < h$ for any $h > 0$.

First of all $\lim_{n\to\infty}\left(\frac{1-n}{1+n}\right) = \lim_{n\to\infty}\frac{\frac{1}{n}-1}{\frac{1}{n}+1} = -1$

Now, for any $h > 0$, $n > \frac{2}{h} - 1$,

$$n+1 > \frac{2}{h} \longrightarrow \frac{n+1}{2} > \frac{1}{h} \rightarrow h > \frac{2}{n+1};$$

$$\left|a_n - (-1)\right| = \left|\frac{1-n}{1+n} + 1\right| = \left|\frac{1-n+1+n}{1+n}\right|$$

$$= \frac{2}{1+n} < h.$$

I'm not sure how this proves $\lim_{n\to\infty}\frac{1-n}{1+n} = -1$

(48) Prove that a convergent sequence a_n cannot have two distinct limits A and B.

[HINT: $|A-B| = |(A-a_n)+(a_n-B)|$

$$\leq |A-a_n| + |a_n - B| \;]$$

Since $\lim_{n\to\infty} a_n = A$, for any $h > 0$, there exists an N_1 such that $|A-a_n| < h$, when $n \geq N_1$.

Likewise $|a_n - B| < h$ for $n \geq N_2$.

For $n \geq N_1$ and N_2 $|A - a_n| + |a_n - B| < 2h$.

$$|A - B| = |(A - a_n) + (a_n - B)| \leq |A - a_n| + |a_n - B| < 2h$$

Since h can be chosen very small for very large n, this means that the limit of $A - B$ as $n \to \infty$ is 0.

$A - B = 0$. $A = B$.

3-7 INFINITE GEOMETRIC SERIES

{1, 4} Write the decimal equivalent of each fraction, using the bar symbol to indicate a repeating digit or block of digits.

① $\dfrac{16}{45}$:

$$
\begin{array}{r}
0.355 \\
45\,\overline{)\,16.000} \\
\underline{13\,5} \\
2\,50 \\
\underline{2\,25} \\
250 \\
\underline{225}
\end{array}
$$

$\therefore \dfrac{16}{45} = 0.3\overline{5}$

② $\dfrac{3}{7}$:

$$
\begin{array}{r}
0.42857142857142857 14 \\
7\,\overline{)\,3.00000000000000}
\end{array}
$$

28, 20, 14, 60, 56, 40, 35, 50, 49, 10 | 10, 7, 30, 28, 20, 14, 60, 56, 40, 35, 50 | 50, 49, 10, 7, 30, 28

$\therefore \dfrac{3}{7} = 0.\overline{428571}$

(3) $\frac{5}{7}$:

$$
\begin{array}{r}
0.714285714 \\
7 \overline{)\,5.000000000}
\end{array}
$$

49
10
7
30
28
20
14
60

60
56
40
35
50
49
10
7
30
28

$\therefore \frac{5}{7} = 0.\overline{714285}$

(4) $\frac{157}{300}$:

$$
\begin{array}{r}
0.5233 \\
300 \overline{)\,157.0000}
\end{array}
$$

1500
700
600
1000
900
1000
900

$\therefore \frac{157}{300} = 0.52\overline{3}$

{5.8} Find the sum of the infinite geometric series for which

(5) $a = 3$, $r = \frac{1}{2}$; $|r| < 1$ so $S_n = \frac{a}{1-r}$

$$\lim_{n\to\infty} S_n = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6$$

(6) $a = -4$, $r = -\frac{1}{3}$; $|r| < 1$

$$\lim_{n\to\infty} S_n = \frac{-4}{1-(-\frac{1}{3})} = \frac{-4}{1+\frac{1}{3}} = \frac{-4}{\frac{4}{3}} = -3$$

(7) $a = 2, \ r = \frac{\sqrt{2}}{2} \approx 0.7071; \ |r| < 1$

$$\lim_{n\to\infty} S_n = \frac{2}{1-\frac{\sqrt{2}}{2}} = \frac{2}{\frac{2-\sqrt{2}}{2}} = \frac{4}{2-\sqrt{2}}$$

To remove the radical from the denominator,

$$\frac{4}{2-\sqrt{2}} \cdot \frac{(2+\sqrt{2})}{(2+\sqrt{2})} = \frac{4(2+\sqrt{2})}{4-2} = 2(2+\sqrt{2})$$

(8) $a = \sqrt{3}, \ r = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}; \ |r| < 1$

$$\lim_{n\to\infty} S_n = \frac{\sqrt{3}}{1-\frac{\sqrt{3}}{3}} = \frac{\sqrt{3}}{\frac{3-\sqrt{3}}{3}} = \frac{3\sqrt{3}}{3-\sqrt{3}}$$

$$\frac{3\sqrt{3}}{3-\sqrt{3}} \ \frac{(3+\sqrt{3})}{(3+\sqrt{3})} = \frac{3\sqrt{3}(3+\sqrt{3})}{9-3} = \frac{9\sqrt{3}+9}{6}$$

$$= \frac{3\sqrt{3}+3}{2} = \frac{3}{2}(\sqrt{3}+1)$$

{9,, 12} Write the first three terms of the infinite geometric progression for which

(9) $a = \frac{2}{3}, \ S = \frac{4}{3}$

$$\frac{4}{3} = \frac{\frac{2}{3}}{1-r} \rightarrow 1-r = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2} \rightarrow 1-\frac{1}{2} = r = \frac{1}{2}$$

$$\frac{2}{3}, \ \frac{2}{3}\left(\frac{1}{2}\right), \ \frac{2}{3}\left(\frac{1}{2}\right)^2 \longrightarrow \frac{2}{3}, \frac{1}{3}, \frac{1}{6}$$

(10) $a = 1.5 = \frac{3}{2}$, $S = 1$

$1 = \dfrac{\frac{3}{2}}{1-r} \rightarrow 1 - r = \frac{3}{2} \rightarrow r = 1 - \frac{3}{2} = -\frac{1}{2}$

$\frac{3}{2}, \frac{3}{2}\left(-\frac{1}{2}\right), \frac{3}{2}\left(-\frac{1}{2}\right)^2 \rightarrow \frac{3}{2}, -\frac{3}{4}, \frac{3}{8}$

(11) $r = -\frac{1}{3}$, $S = \frac{3}{8}$; $\frac{3}{8} = \dfrac{a}{1+\frac{1}{3}} = \dfrac{a}{\frac{4}{3}}$

$a = \frac{3}{8} \cdot \frac{4}{3} = \frac{1}{2}$ $\quad \frac{1}{2}, \frac{1}{2}\left(-\frac{1}{3}\right), \frac{1}{2}\left(\frac{1}{3}\right)^2$

$\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}$

(12) $r = 0.01$, $S = \frac{3}{11}$; $r = \frac{1}{100}$

$\frac{3}{11} = \dfrac{a}{1 - \frac{1}{100}} = \dfrac{a}{\frac{100-1}{100}} = \dfrac{a}{\frac{99}{100}}$

$a = \frac{3}{11} \cdot \frac{99}{100} = \frac{27}{100}$

$\frac{27}{100}, \frac{27}{100}\left(\frac{1}{100}\right), \frac{27}{100}\left(\frac{1}{100}\right)^2 \rightarrow \frac{27}{100}, \frac{27}{10000}, \frac{27}{1000000}$

{13..16} Find the sum of the infinite geometric series.

(13) $18 + 12 + 8 + \ldots$

First find $r = \frac{12}{18} = \frac{2}{3}$ and $\frac{8}{12} = \frac{2}{3}$ so $r = \frac{2}{3}$

$a = 18$

$S_n = \dfrac{18}{1 - \frac{2}{3}} = \dfrac{18}{\frac{1}{3}} = 3 \cdot 18 = 54$

(14) $\sqrt{3} + \sqrt{\frac{3}{2}} + \frac{1}{2}\sqrt{3} + \dots$

$\frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$ $\quad r = \frac{\frac{\sqrt{6}}{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{18}}{2 \cdot 3} = \frac{3\sqrt{2}}{2 \cdot 3} = \frac{\sqrt{2}}{2}$

Also $\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{6}}{2}} = \frac{\sqrt{3}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{18}}{6} = \frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2}$

So, $r = \frac{\sqrt{2}}{2}$ and $S_n = \frac{\sqrt{3}}{1 - \frac{\sqrt{2}}{2}} = \frac{\sqrt{3}}{\frac{2-\sqrt{2}}{2}}$

$= \frac{2\sqrt{3}}{2-\sqrt{2}} \frac{(2+\sqrt{2})}{(2+\sqrt{2})} = \frac{2\sqrt{3}(2+\sqrt{2})}{4-2} = \frac{4\sqrt{3}+2\sqrt{6}}{2}$

$= \boxed{2\sqrt{3}+\sqrt{6}} = \boxed{\sqrt{3}(2+\sqrt{2})}$

(15) $0.2 - 0.02 + 0.002 - \dots$

$a = 0.2$, $r = -0.1$ so $S_n = \frac{0.2}{1-(-0.1)} = \frac{0.2}{1.1} = \frac{2}{11}$

(16) $3 + 3(10)^{-1} + 3(10)^{-2} + \dots$

obviously $r = 10^{-1} = \frac{1}{10}$

$S_n = \frac{3}{1-\frac{1}{10}} = \frac{3}{\frac{9}{10}} = \frac{30}{9} = \frac{10}{3}$

(17) Find the indicated sum.

(17) $\sum_{k=0}^{\infty} 3\left(\frac{1}{2}\right)^k = \lim_{n \to \infty} S_n = \frac{3}{1-\frac{1}{2}} = 6$

(18) $$\sum_{k=0}^{\infty} 2\left(-\frac{1}{3}\right)^k = \lim_{n\to\infty} S_n = \frac{2}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{2}{1+\frac{1}{3}} = \frac{2}{4/3} = 2\cdot\frac{3}{4} = \frac{3}{2}$$

Write the given repeating decimal as an equivalent common fraction.

(19) $$0.\overline{4} = \frac{4}{10^1} + \frac{4}{10^2} + \frac{4}{10^3} + \dots$$

$a = 0.4$, $r = \frac{1}{10}$

$$S_n = \frac{0.4}{1-0.1} = \frac{0.4}{0.9} = \frac{4}{9}$$

(20) $$0.\overline{324} = \frac{324}{10^3} + \frac{324}{10^6} + \frac{324}{10^9} + \dots$$

so, $a = 0.324$, $r = 10^{-3} = 0.001$

$$S_n = \frac{0.324}{1-0.001} = \frac{0.324}{0.999} = \frac{324}{999} = \frac{12}{37}$$

(21) $1.\overline{24}$ (22) $2.1\overline{3}$

(21)

$$1.\overline{24} = 1 + \left[\frac{24}{10^2} + \frac{24}{10^4} + \frac{24}{10^6} + \ldots\right]$$

so, $a = 0.24$, $r = 10^{-2} = 0.01$

$$S_n = 1 + \frac{0.24}{1-0.01} = 1 + \frac{0.24}{0.99} = 1 + \frac{8}{33} = \frac{41}{33}$$

(22)

$$2.1\overline{3} = 2 + \frac{1}{10} + \left[\frac{3}{10^2} + \frac{3}{10^3} + \ldots\right]$$

$$2.1\overline{3} = 2.1 + \left[\frac{3}{10^2} + \frac{3}{10^3} + \ldots\right]$$

$a = 0.03$, $r = 0.1$

$$2.1\overline{3} = 2.1 + \frac{0.03}{1-0.1} = 2.1 + \frac{0.03}{0.9}$$

$$= 2.1 + \frac{3}{90} = \frac{21}{10} + \frac{1}{30} = \frac{21(3)+1}{30} = \frac{64}{30} = \frac{32}{15}$$

{23, 28} Support your answer with an explanation.

(23) If two infinite geometric series have the same finite sum, are they the same series?

Not necessarily; given $\frac{a_1}{1-r_1} = \frac{a_2}{1-r_2}$,

we cannot necessarily conclude that $a_1 = a_2$ and $r_1 = r_2$

ent

$= \frac{12}{37}$

(24) Can the sum of a convergent infinite geometric series be less than the first term?

For an infinite geometric series to be convergent, the absolute value of common ratio $|r| < 1$
If $r < 0$, then the sum can be less than the first term, yes.

That is $\frac{a}{1-r} < a$ if and only if $r < 1$

($|r| < 1$ for a convergent series)

and $1 < 1 - r \rightarrow r < 0$

(25) Can the series $\sum_{i=1}^{\infty} a$ be convergent for any $a \in \mathbb{R}$?

$\sum_{i=1}^{\infty} a = \lim_{n \to \infty} a \cdot n$ is convergent only for $a = 0$.

(26) Given an infinite geometric series in which $a > 0$ and $|r| < 1$. Can the sum of this series be negative?

NO; $S = \frac{a}{1-r}$, $|r| < 1$, $a > 0$.

$1 - r > 0$ for $|r| < 1$ $\therefore$ $S > 0$.

(27) Is there an infinite geometric series for which $a = 6$ and $S = \frac{2}{3}$?

$\frac{2}{3} = \frac{6}{1-r} \longrightarrow 2 - 2r = 18 \longrightarrow -2r = 16$

$r = -8$, Since $|r| = |-8| \geq 1$, the series diverges and the sum is not defined.

(28) Given the statement: Every repeating decimal represents a rational number. Is the converse of this statement true?

CONVERSE: Every rational number can be represented by a repeating decimal.

YES. When you divide an integer r by a positive integer s, the remainder at each step belongs to $\{0, 1, 2, \ldots, s-1\}$. Thus, within $s-1$ steps after only zeros

are left in the dividend, either zero occurs as a remainder and the division process stops, or a nonzero remainder recurs, and the process thereafter produces a repeating sequence of remainders, with a repeating block of digits in the quotient.

(29) An infinite geometric series has the sum 8. If the sum of the first two terms is 2, find a and r.

$$S = 8 = \frac{a}{1-r} \rightarrow 8 - 8r = a$$

$$8 - a = 8r \rightarrow r = \frac{8-a}{8}$$

$$a + ar = 2 \rightarrow a + a\left(\frac{8-a}{8}\right) = 2$$

$$a + \frac{8a - a^2}{8} = 2 \rightarrow 8a + 8a - a^2 = 16$$

$$a^2 - 16a + 16 = 0 \rightarrow (a^2 - 16a + (-8)^2) = -16 + 64$$

$$(a-8)^2 = 48 \rightarrow a - 8 = \pm\sqrt{48} = \pm 4\sqrt{3}$$

$$a = 8 + 4\sqrt{3} \text{ or } a = 8 - 4\sqrt{3}$$

$$r = 1 - \frac{a}{8} = 1 - \frac{8-4\sqrt{3}}{8}$$

$$r = \frac{8-(8-4\sqrt{3})}{8} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

or $$r = \frac{8-(8+4\sqrt{3})}{8} = \frac{-4\sqrt{3}}{8} = -\frac{\sqrt{3}}{2}$$

$$a + ar = (8+4\sqrt{3}) + (8+4\sqrt{3})\left(-\frac{\sqrt{3}}{2}\right)$$
$$= 8 + 4\sqrt{3} - 4\sqrt{3} - 6 = 2$$

$$a + ar = (8-4\sqrt{3}) + (8-4\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)$$
$$= 8 - 4\sqrt{3} + 4\sqrt{3} - 6 = 2$$

so $r = \pm\frac{\sqrt{3}}{2}$ and $a = 4(2 \pm \sqrt{3})$

(30) An infinite geometric series has 2 as its sum. If the second term is $\frac{3}{8}$, find a and r.

$$S = 2 = \frac{a}{1-r} \longrightarrow 2 - 2r = a \rightarrow 2r = 2 - a$$

$$r = 1 - \frac{a}{2}; \quad a \cdot r = \frac{3}{8} \longrightarrow a\left(1 - \frac{a}{2}\right) = \frac{3}{8}$$

$$a - \frac{a^2}{2} = \frac{3}{8} \longrightarrow 2a - a^2 = \frac{3}{4}$$

$$8a - 4a^2 = 3 \longrightarrow 4a^2 - 8a + 3 = 0$$

$(4)(3) = 12 = u \cdot v$ where $u + v = -8$; FACTORS OF 12: $(-1)(-12)$, $\boxed{(-2)(-6)}$

$(-2)(-6) = 12$; $(-2)+(-6) = -8$; $4a^2 - 2a - 6a + 3 = 2a(2a-1) - 3(2a-1)$

$= (2a-1)(2a-3) = 0$ when $2a = 1 \rightarrow a = \frac{1}{2}$ or $2a = 3 \rightarrow a = \frac{3}{2}$

$a = \frac{1}{2}$ or $a = \frac{3}{2}$

so $r = 1 - \frac{a}{2}$;

when $a = \frac{1}{2}$

$r = 1 - \frac{1/2}{2} = 1 - \frac{1}{4} = \frac{3}{4}$

when $a = \frac{3}{2}$, $r = 1 - \frac{3/2}{2} = 1 - \frac{3}{4} = \frac{1}{4}$

so, either $a = \frac{1}{2}$ and $r = \frac{3}{4}$

or $a = \frac{3}{2}$ and $r = \frac{1}{4}$

{31, 32} Express the exact error involved in each of the indicated approximations.

(31) $\frac{8}{3} \approx 2.67$

$$\begin{array}{r} 2.666 \\ 3\overline{)8.0000} \\ \underline{6} \\ 2\,0 \\ \underline{1\,8} \\ 20 \\ \underline{18} \\ 20 \end{array}$$

error $= 0.00\overline{33333}$

NO, THAT'S NOT $\boxed{\text{EXACT}}$

$2.67 = \frac{267}{100} = \frac{801}{300}$; $\frac{8}{3} = \frac{800}{300}$

$$\left| 2.67 - \frac{8}{3} \right| = \left| \frac{801}{300} - \frac{800}{300} \right| = \frac{1}{300}$$

$\frac{1}{300}$ is the exact error.

(32) $3\frac{7}{11} \approx 3.65 \longrightarrow \frac{40}{11} \approx 3.65$

$$3.65 = \frac{365}{100} = \frac{4015}{1100} \;;\; \frac{40}{11} = \frac{4000}{1100}$$

$$\left|3.65 - \frac{40}{11}\right| = \left|\frac{4015}{1100} - \frac{4000}{1100}\right| = \frac{15}{1100} = \frac{3}{220}$$

{33,..,36} Find the range of values of x for which the sum of each geometric progression can be obtained.

(33) $\frac{3}{5}, \frac{3}{5}(x-1), \frac{3}{5}(x-1)^2, \frac{3}{5}(x-1)^3$

$|x-1| < 1 \longrightarrow -1 < x-1 < 1$

$0 < x < 2$

(34) $1, 2(x-2), 4(x-2)^2, 8(x-2)^3, \ldots$

$1, 2^1(x-2)^1, 2^2(x-2)^2, 2^3(x-2)^3, \ldots$

$r = 2(x-2) \rightarrow |r| < 1$

$|2(x-2)| < 1 \rightarrow -1 < 2x-4 < 1$

$3 < 2x < 5$

$\frac{3}{2} < x < \frac{5}{2}$

(35) $1,\ 3(4-x),\ 9(4-x)^2,\ 27(4-x)^3, \ldots$ (38)

$r = 3(4-x)$ so $|3(4-x)| < 1$

$-1 < 12-3x < 1$

$-13 < -3x < -11$

$\frac{-13}{-3} > x > \frac{-11}{-3} \rightarrow \frac{11}{3} < x < \frac{13}{3}$

(36) $\frac{1}{2},\ \frac{1}{4}(2-x),\ \frac{1}{8}(2-x)^2,\ \frac{1}{16}(2-x)^3, \ldots$

$r = \frac{1}{2}(2-x)$ and $|r| < 1$

so, $\left|\frac{1}{2}(2-x)\right| < 1$

$-1 < 1-\frac{x}{2} < 1 \rightarrow -2 < -\frac{x}{2} < 0$

$(-2)(-2) > x > 0 \rightarrow 0 < x < 4$

(37) Find r for the infinite geometric series in which

$S = \frac{4+3\sqrt{2}}{2}$ and $a = \sqrt{2}+1$

$\frac{4+3\sqrt{2}}{2} = \frac{\sqrt{2}+1}{1-r} \rightarrow (4+3\sqrt{2})(1-r) = 2\sqrt{2}+2$

$4-4r+3\sqrt{2}-3r\sqrt{2} = 2\sqrt{2}+2$

$r(4+3\sqrt{2}) = 2+\sqrt{2}$

$r = \frac{2+\sqrt{2}}{4+3\sqrt{2}} \cdot \frac{(4-3\sqrt{2})}{(4-3\sqrt{2})} = \frac{8-6\sqrt{2}+4\sqrt{2}-6}{16-18} = \frac{2-2\sqrt{2}}{-2}$

$= \sqrt{2}-1$

(38) The sum of the first two terms of an infinite geometric series is 2, the sum of the first three terms is 3, and $r \neq 1$. Show that this series is convergent.

$a + a \cdot r = 2$ and $a + ar + ar^2 = 3$

$a \cdot r = 2 - a$

$r = \frac{2}{a} - 1$

$a + a\left(\frac{2}{a} - 1\right) + a\left(\frac{2}{a} - 1\right)^2 = 3$

$a + 2 - a + a\left[\frac{4}{a^2} - \frac{4}{a} + 1\right] = 3$

$2 + \frac{4}{a} - 4 + a = 3 \longrightarrow -2 + \frac{4}{a} + a = 3$

$\left(a + \frac{4}{a}\right) = 5 \longrightarrow \frac{a^2 + 4}{a} = 5 \longrightarrow a^2 + 4 = 5a$

$a^2 - 5a + 4 = 0 \rightarrow (a-1)(a-4) = 0$

$a = 1$ or $a = 4$

so $r = \frac{2}{1} - 1 = 1$ or $r = \frac{2}{4} - 1 = -\frac{1}{2}$

when $a = 4$, $\left|-\frac{1}{2}\right| < 1$

but when $a = 1$, $|1| \not< 1$.

$a = 4$, $r = -\frac{1}{2}$ for a convergent series.

$\frac{4}{1 + \frac{1}{2}} = \frac{4}{\frac{3}{2}} = \frac{8}{3}$

{39..42} Solve for x, given that each infinite geometric series converges to the indicated limit. 43

(39) $\frac{3}{4} = 1 + 2x + 4x^2 + \ldots$ $a = 1$, $r = 2x$

$S = \frac{3}{4} = \frac{1}{1-2x} \rightarrow 1 - 2x = \frac{4}{3} \rightarrow 3 - 6x = 4$

$-6x = 1$

$x = -1/6$

(40) $\frac{6}{7} = 1 + 3x + 9x^2 + \ldots$ $a = 1$, $r = 3x$

$S = \frac{6}{7} = \frac{1}{1-3x} \rightarrow 1 - 3x = \frac{7}{6}$

$6 - 18x = 7 \rightarrow 18x = -1 \rightarrow x = -\frac{1}{18}$

(41) $\frac{8x}{4x-1} = x + x^2 + x^3 + \ldots$ $a = x$, $r = x$ 44

$S = \frac{8x}{4x-1} = \frac{x}{1-x} \rightarrow 8x - 8x^2 = 4x^2 - x$

$12x^2 - 9x = 0 \rightarrow 3x(4x-3) = 0$

$x = 0$ or $4x = 3 \rightarrow x = 3/4$

(42) $\frac{2x}{x+1} = x + x^2 + x^3 + \ldots$ $a = x$, $r = x$

$S = \frac{2x}{x+1} = \frac{x}{1-x} \rightarrow 2x - 2x^2 = x^2 + x$

$x = 0$ or $3x = 1$

$3x^2 - x = 0 \rightarrow x(3x-1) = 0$; $x = 1/3$

(43) A ball is dropped from a height of 32 ft. Each time it strikes the ground, it rebounds $\frac{3}{8}$ ths of the distance from which it had fallen. Theoretically, how far will the ball travel before coming to rest?

distance traveled : $32 + 2\left[32\left(\frac{3}{8}\right)\right] + 2\left[32\left(\frac{3}{8}\right)^2\right] + 2\left[32\left(\frac{3}{8}\right)^3\right] + 2\left[32\left(\frac{3}{8}\right)^4\right] + \ldots$

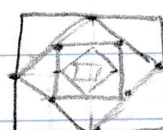
$$= 32 + 2(32)\underbrace{\left[\left(\frac{3}{8}\right) + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^3 + \ldots + \left(\frac{3}{8}\right)^n + \ldots\right]}$$

$$S = \sum_{k=1}^{\infty}\left(\frac{3}{8}\right)^k = \frac{\frac{3}{8}}{1-\frac{3}{8}} = \frac{3}{5}$$

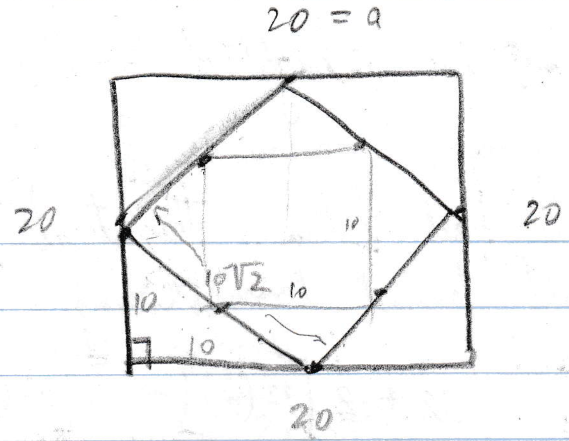
$$= 32 + 2(32)\left(\frac{3}{5}\right)$$

$$= 32 + \frac{192}{5} = \frac{160+192}{5} = \frac{352}{5} = 70.4 \text{ feet}$$

(44) Given a square of side 20 in.
The midpoints of its sides are joined to form an inscribed square.
The midpoints of this second square are joined to form a third square.



If the process is continued endlessly, find the sum of the perimeters and the sum of the areas of all the squares, including the initial one.



$$\sqrt{10^2+10^2} = \sqrt{200} = 10\sqrt{2}$$

$$[4 \cdot 20 = 80] + [4 \cdot 10\sqrt{2} = 40\sqrt{2}] + [4 \cdot 10 = 40] +$$

$$\sqrt{(5\sqrt{2})^2 + (5\sqrt{2})^2} = \sqrt{25 \cdot 2 + 25 \cdot 2} = \sqrt{100} = 10$$

$$80 \cdot \frac{1}{\sqrt{2}} = \frac{80}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{80\sqrt{2}}{2} = 40\sqrt{2}$$

$$(40\sqrt{2}) \cdot \frac{1}{\sqrt{2}} = 40$$

For sum of perimeters : $a = 80$, $r = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$$S = \frac{80}{1 - \frac{\sqrt{2}}{2}} = \frac{80}{\frac{2-\sqrt{2}}{2}} = \frac{160}{2-\sqrt{2}}$$

$$\frac{160}{2-\sqrt{2}} \frac{(2+\sqrt{2})}{(2+\sqrt{2})} = \frac{320 + 160\sqrt{2}}{4-2} = 160 + 80\sqrt{2}$$

$$= 80(2+\sqrt{2})$$

For areas: $20^2 + (10\sqrt{2})^2 + 10^2 + \ldots$

$$\frac{(10\sqrt{2})^2}{20^2} = \frac{200}{400} = \frac{1}{2} \quad \text{and} \quad \frac{10^2}{(10\sqrt{2})^2} = \frac{100}{200} = \frac{1}{2}$$

Sum of areas: $a = 400$, $r = \frac{1}{2}$

$$S = \frac{400}{1-\frac{1}{2}} = \frac{400}{\frac{1}{2}} = 800 \text{ ft}^2$$

I came up with the correct result THINKING MATHEMATICALLY, using the theorem of Pythagora.

alternately, the diagonal of a square of side s has length $s\sqrt{2}$.

Sides of successive squares in this problem are diagonals of squares with sides half the length of the preceeding square, $\frac{s}{2}\sqrt{2}$

$\therefore$ Perimeter $= 4(20) + \frac{\sqrt{2}}{2}\cdot 4(20) + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\cdot 4(20)\right) + \ldots$

$\therefore a = 80$, $r = \frac{\sqrt{2}}{2}$

I concluded the same thing with brute force, deriving the common ratio with first four terms.

Area $= (20)^2 + \left(\frac{\sqrt{2}}{2}\cdot 20\right)^2 + \left(\frac{\sqrt{2}}{2}\cdot 20\right)^3 + \ldots$

$a = 400$, $r = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$

(45) Given the infinite geometric series

$$a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \dots + a \cdot r^{n-1} \dots \quad (a > 0)$$

1 2 3 4

If $0 < r < 1$, what fractional part of the sum is the sum of the odd-numbered terms? the even-numbered terms?

How do we know which terms are odd and which terms are even? The question is not asking about whether the value of each term is odd or even. It is referring to first, third, fifth, ... terms versus second, fourth, sixth, ... terms.

even + even → even
odd + odd → odd
odd + even → even

(46)

So, $S_0 = a + ar^2 + ar^4 + ar^6 + \dots$

where $a = a$ and $r = r^2$, so

$$S_0 = \frac{a}{1-r^2}$$

and $S_E = (a \cdot r) + (a \cdot r) r^2 + (a \cdot r^3) r^2 + (a \cdot r^5) r^2$

where "a" $= ar$ and "r" $= r^2$

$$S_E = \frac{ar}{1-r^2};$$

What fractional part of $S = \frac{a}{1-r}$ is S_0? $\frac{S_0}{S}$

$$\frac{S_o}{S} = \frac{\frac{a}{1-r^2}}{\frac{a}{1-r}} = \frac{1-r}{1-r^2} = \frac{1-r}{(1+r)(1-r)} = \frac{1}{1+r}$$

What fractional part of $S = \frac{a}{1-r}$ is S_E?

$$\frac{S_E}{S} = \frac{\frac{a \cdot r}{1-r^2}}{\frac{a}{1-r}} = \frac{a \cdot r}{1-r^2} \cdot \frac{1-r}{a} = \frac{a \cdot r}{(1+r)(1-r)} \cdot \frac{(1-r)}{a}$$

$$= \frac{r}{1+r}$$

---

(46) Given the series $a + \sum_{n=1}^{\infty} a \cdot r^n$, in which $|r| < 1$. If each term in the series is k times the sum of all the terms that follow it $(k > 0)$, express the series in terms of a and k.

Solution: For a given n $\sum_{k=n+1}^{\infty} a \cdot r^{k-1} = S - S_n$

$$= \frac{a}{1-r} - \frac{a - a \cdot r^n}{1-r} = \frac{a \cdot r^n}{1-r};$$

$$a_n = a \cdot r^{n-1} = \frac{k \cdot a r^n}{1-r};$$

$$1 = \frac{k \cdot r}{1-r} \rightarrow 1 - r = k \cdot r \rightarrow r(k+1) = 1$$

$$\rightarrow r = \frac{1}{k+1}$$

Let's think this through.

Given series $a + \sum_{n=1}^{\infty} a \cdot r^n$, $|r| < 1$

Each term in the series is k times the sum of all the terms that follow it. Express the series in terms of a and k.

First we want to express r in terms of k. Then we just substitute that expression for r.

We know $\boxed{S = \frac{a}{1-r}}$

Each term $\boxed{a_n = a \cdot r^{n-1}}$

The sum of all the terms following S_n is $S - S_n$

That is, for a given n, $\sum_{k=n+1}^{\infty} a \cdot r^{k-1} = S - S_n$

What is S_n? SEE P. 85 in TEXT $\boxed{S_n = \frac{a - a \cdot r^n}{1-r}}$

So, the sum of all the terms following S_n is $S - S_n = \frac{a}{1-r} - \frac{a - a \cdot r^n}{1-r} = \boxed{\frac{a \cdot r^n}{1-r}}$

47

We are also told that each term in the series is k times the sum of all the terms that follow it.

In symbols, $a_n = a_1 r^{n-1} = k\left[\dfrac{a_1 r^n}{1-r}\right]$

$$a_1 r^{n-1} = \frac{k \cdot a_1 r^n}{1-r}$$

$$\frac{a_1 r^{n-1}}{a_1 r^{n-1}} = \frac{1}{a_1 r^{n-1}} \cdot \frac{k \cdot a_1 r^n}{1-r} \longrightarrow 1 = \frac{kr}{1-r}$$

$$1 - r = k \cdot r \longrightarrow 1 = kr + r = r(k+1)$$

$r = \dfrac{1}{k+1}$, so we can express the series in terms of a and k:

$$a + \sum_{n=1}^{\infty} a_1 r^n = a + \sum_{n=1}^{\infty} a \cdot \left(\frac{1}{k+1}\right)^n$$

---

(47) Prove the theorem on page 104.

"The sum of any convergent infinite geometric series whose first term and common ratio are rational numbers is a rational number."

$S = \dfrac{a}{1-r}$; $\dfrac{p}{q} = a$, $\dfrac{s}{t} = r$, where $p, q, s, t \in \mathbb{Z}$

Since $p, q, s, t \in \{\text{integers}\}$, a and r are rational numbers. That is, $a, r \in \mathbb{Q}$.

$$S = \frac{a}{1-r} = \frac{\frac{p}{q}}{1-\frac{s}{t}} = \frac{\frac{p}{q}}{\frac{t-s}{t}} = \frac{p \cdot t}{q(t-s)},$$

$(q(t-s) \neq 0)$.

Since the integers are closed under addition and multiplication, S is the ratio of two integers and is therefore rational.

---

(48) Prove that every repeating decimal represents a rational number.

Let $b + .\overline{a_1 a_2 a_3 \ldots a_n}$ be a repeating decimal where b is an integer and each a_k an integer between 0 and 9.

This number forms a geometric sequence where $a = (a_1 a_2 a_3 \ldots a_n)(10^{-n})$ and $r = 10^{-n}$.

Then $S = b + \dfrac{(a_1 a_2 a_3 \ldots a_n)\, 10^{-n}}{1-10^{-n}} = b + \dfrac{a_1 a_2 \ldots a_n}{10^n - 1}$

$$= \frac{b(10^n - 1) + a_1 a_2 \cdots a_n}{10^n - 1}$$

which is the ratio of two integers and so is rational.

3-8 AXIOM OF COMPLETENESS

(1) Does the axiom of completeness permit you to conclude that a sequence which does not have a limit cannot be a nondecreasing sequence?

No. An unbounded sequence which has no limit may be nondecreasing.

(2) Explain why a sequence that is not bounded cannot converge.

By the definition of convergence, all the terms beyond a certain point of a sequence remain arbitrarily close to some finite number. This they cannot do if they are unbounded.

(3) If $\sum_{k=1}^{\infty} t_k$ is a convergent series, is $\sum_{k=1}^{\infty} (t_k + 1)$ also a convergent series? NO because

$$\sum_{k=1}^{\infty} (t_k + 1) = \sum_{k=1}^{\infty} t_k + \sum_{k=1}^{\infty} 1 = L + \lim_{n \to \infty} n,$$

which is unbounded.

(4) Can the product of two repeating decimals be a nonrepeating, nonterminating decimal? Give a reason in support of your answer. (7)

No because the 2 repeating decimals represent rational numbers and the product of two rationals is rational, which in turn has a repeating decimal representation. (8)

{5..10} Classify the given statements as true or false.
For each you classify as false, provide a counterexample.

(5) The sum of an infinite series is the limit of a sequence of partial sums. (9)

TRUE (10)

(6) The partial sums of a series must increase without bound as n increases.

FALSE The sums may decrease without bound, increase without bound, or may fluctuate.

(7) A sequence of partial sums $S_1, S_2, \ldots, S_n, \ldots$ is nondecreasing if S_n never decreases as n increases. TRUE

(8) If each odd-numbered term of a series is greater than zero and each even-numbered term is less than zero, the series cannot be convergent. FALSE

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n}; \quad a = \frac{1}{2}, \quad r = \left(-\frac{1}{2}\right)$$

$$S = \lim_{n \to \infty} S_n = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}$$

(9) In the series $\sum_{n=1}^{\infty} a_n$, $S_n - S_{n-1} = a_n$ for $n > 1$.

TRUE

(10) Every nondecreasing sequence of partial sums is bounded. FALSE

$$\sum_{k=1}^{\infty} k$$

{11..14} Write the first four terms of the sequence of partial sums for the given series and tell whether the sequence is nondecreasing.

(11) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \left(\frac{1}{2}\right)^n + \dots$

$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$,

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$, $\dots$, $S_n = \frac{n^2+1}{n^2}$

$\lim_{n\to\infty} S_n = 1$; $\frac{1}{2} < \frac{3}{4} < \frac{7}{8} < \frac{15}{16}$ ∴ nondecreasing

(12) $1 - 2 + 3 - 4 + \dots (-1)^{n-1} \cdot n + \dots$

$S_1 = 1$, $S_2 = 1 - 2 = -1$, $S_3 = 1 - 2 + 3 = 2$,

$S_4 = 1 - 2 + 3 - 4 = -2$, $\dots$ no, not nondecreasing.

(13) $\left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \dots + \left(\frac{1}{n+1} - \frac{1}{n}\right) + \dots$

$S_1 = -\frac{1}{2}$, $S_2 = \left(-\frac{1}{2}\right) + \left(-\frac{1}{6}\right) = -\frac{2}{3}$,

$S_3 = -\frac{1}{2} - \frac{1}{6} + \left(\frac{1}{4} - \frac{1}{3}\right) = -\frac{2}{3} - \frac{1}{12} = -\frac{3}{4}$

$S_4 = -\frac{1}{2} - \frac{1}{6} - \frac{1}{12} + \left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{3}{4} - \frac{1}{20} = -\frac{16}{20}$

$= -\frac{4}{5}$; $-\frac{1}{2} > -\frac{2}{3} > -\frac{3}{4} > -\frac{4}{5}$; NO, not nondecreasing

(14) $\frac{3}{2} + \frac{5}{4} + \frac{9}{8} + \dots + \frac{2^n+1}{2^n} + \dots$

$S_1 = \frac{3}{2}$, $S_2 = \frac{3}{2} + \frac{5}{4} = \frac{11}{4}$, $S_3 = \frac{3}{2} + \frac{5}{4} + \frac{9}{8} = \frac{11}{4} + \frac{9}{8}$

$= \frac{31}{8}$, $S_4 = \frac{3}{2} + \frac{5}{4} + \frac{9}{8} + \frac{17}{16} = \frac{31}{8} + \frac{17}{16} = \frac{79}{16}$

$\frac{3}{2} < \frac{11}{4} < \frac{31}{8} < \frac{79}{16}$; yes, nondecreasing

(15) Given the infinite geometric series $\sum_{n=0}^{\infty} ar^n$ with $a > 0$. For the indicated value of r tell whether the series is convergent or divergent.

(a) $r = 1$ (b) $r = \frac{1}{2}$ (c) $r = -\frac{1}{3}$ (d) $r = -\frac{3}{2}$

For geometric series, when $|r| < 1$, convergent
When $|r| \geq 1$, the series diverges.

(a) $|1| = 1$ so divergent
(b) $|\frac{1}{2}| < 1$ so convergent
(c) $|-\frac{1}{3}| < 1$ so convergent
(d) $|-\frac{3}{2}| > 1$ so divergent

(16) Given the series $\sum_{n=1}^{\infty} t_n$, in which all terms are positive. What conclusions, if any, can you make about its convergence or divergence if for all n (a) $t_n < \frac{1}{2}$?

(b) $t_n > \frac{1}{2}$? (c) $\lim_{n\to\infty} t_n = 1$?

(d) $\lim_{n\to\infty} t_n < 0$?

$$\sum_{n=1}^{\infty} t_n \ ; \quad \begin{array}{l} (a)\ t_n < \frac{1}{2} \\ (b)\ t_n > \frac{1}{2} \\ (c)\ \lim_{n\to\infty} t_n = 1 \\ (d)\ \lim_{n\to\infty} t_n < 0 \end{array}$$

(a) No conclusions for $t_n < \frac{1}{2}$

Series may diverge ($t_n = \frac{1}{4} < \frac{1}{2}$ for all n)

(b) or it may converge ($t_n = \frac{1}{3^n} < \frac{1}{2}$ for all n

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$)

(b) diverges; since $t_n > \frac{1}{2}$ for all n,

$\displaystyle\sum_{n=1}^{\infty} t_n > \sum_{n=1}^{\infty} \frac{1}{2} = \lim_{n\to\infty} \frac{n}{2}$, which

is unbounded. $\therefore \displaystyle\sum_{n=1}^{\infty} t_n$ is unbounded

(c) Series diverges; $S_n = S_{n-1} + t_n$,

$\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} + 1$,

Series is unbounded

(d) None; series may diverge ($t_n = -1 < 0$ for all n), or it may converge. $\left(\sum_{n=1}^{\infty} -\frac{1}{3^n} = \frac{-\frac{1}{3}}{1-\frac{1}{3}} = -\frac{1}{2} \right)$

But all $t_n > 0$

{17, 20} Prove the given series convergent by citing a convergent geometric series whose sum bounds the given series.

(17) $\frac{1}{5} + \frac{1}{5^3} + \frac{1}{5^6} + \dots + \frac{1}{5^{\frac{n(n+1)}{2}}} + \dots$

$$\sum_{n=1}^{\infty} \frac{1}{5^{\frac{n(n+1)}{2}}} \leq \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n \text{ since } \frac{n(n+1)}{2} \geq n$$

$$\therefore \frac{1}{5^{\frac{n(n+1)}{2}}} \leq \frac{1}{5^n} \quad \left\{ \text{is bounded by } \frac{1}{5^n} \right.$$

(18) $\frac{1}{3+1}, \frac{1}{3^2+1}, \frac{1}{3^3+1} + \dots + \frac{1}{3^n+1} + \dots$

$$\sum_{n=1}^{\infty} \frac{1}{3^n+1} \leq \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

(19) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots + \frac{1}{n^n} + \dots$

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} \leq 1 + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$$

(20) $\frac{1}{2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{4} + \ldots + \frac{n}{n+1}\cdot\frac{1}{2^n} + \ldots$

$$\sum_{n=1}^{\infty} \frac{n}{n+1}\cdot\frac{1}{2^n} < \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \text{ since } \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

and $\frac{n}{n+1} < 1$, so $\frac{n}{n+1}\cdot\frac{1}{2^n} < \left(\frac{1}{2}\right)^n$

{21,..,26} The n^{th} partial sum S_n of a series is given. By investigating $\lim_{n\to\infty} S_n$ determine whether the series is convergent or divergent. When convergent, give the sum.

(21) $S_n = \frac{1}{2}n(n+1) = \frac{1}{2}(n^2+n)$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}(n^2+n) = \infty$$

hence, divergent

(22) $S_n = 1 - \frac{1}{2^n}$; $\lim_{n\to\infty} S_n = 1$

$\therefore$ Convergent

(23) $S_n = \frac{n}{2n+1}$; $\lim_{n\to\infty} S_n = \frac{1}{2}$

$\therefore$ Convergent

(24) $S_n = \dfrac{n(2n-1)}{n+3} = \dfrac{2n^2-n}{n+3}$

$\lim_{n\to\infty} S_n = \infty$ $\therefore$ divergent (unbounded)

(25) $S_n = \dfrac{2n+3}{3n+2}$; $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{2+\frac{3}{n}}{3+\frac{2}{n}} = \dfrac{2}{3}$

$\therefore$ convergent

(26) $S_n = \dfrac{n+1}{n^2+n+1}$; $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}} = 0$

$\therefore$ convergent

{27,, 30} The numbers given are the first four partial sums of a series. Form an appropriate series and tell whether it is convergent or divergent.

(27) $18, 24, 26, 26\frac{2}{3}$

$a_1 = 18$

$a_2 = 6$

$a_3 = 26 - 24 = 2$

$a_4 = 26\frac{2}{3} - 26 = \frac{2}{3}$

$r = \frac{1}{3}$

$6 = r(18) \rightarrow r = \frac{6}{18} = \frac{1}{3}$

$24 = \dfrac{18 - 18\left(\frac{1}{3}\right)^2}{1-\frac{1}{3}} = \dfrac{18-2}{\frac{2}{3}} = 16 \cdot \frac{3}{2} = 24$ ✓

$\sum_{n=0}^{\infty} 18\left(\frac{1}{3}\right)^n$; convergent

(28) $1, 3, 6, 10$

$a_1 = 1$

$a_2 = 3-1 = 2$

$a_3 = 6-3 = 3$

$a_4 = 10-6 = 4$

$$\sum_{n=1}^{\infty} n$$

divergent

(29) $\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}$

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$a_3 = \frac{3}{8} - \left(\frac{1}{4}\right) = \frac{3}{8} - \frac{2}{8} = \frac{1}{8}$

$a_4 = \frac{5}{16} - \frac{3}{8} = \frac{5}{16} - \frac{6}{16} = -\frac{1}{16}$

$r = -\frac{1}{2}$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n (-1)^{n+1} \qquad S_n = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

convergent

(30) $1, 0, 1, 0$

$a_1 = 1$

$a_2 = 0-1 = -1$

$a_3 = 1-0 = 1$

$a_4 = 0-1 = -1$

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

divergent

(31) Prove: If $\sum_{n=1}^{\infty} u_n$ converges with a sum S and k is any constant, then $\sum_{n=1}^{\infty} k \cdot u_n$ converges with a sum $k \cdot S$.

Let $S_n = u_1 + u_2 + \dots + u_n$. Then $\lim_{n\to\infty} S_n = S$

Let $T_n = k \cdot u_1 + k \cdot u_2 + \dots + k \cdot u_n$
$= k(u_1 + u_2 + \dots + u_n) = k \cdot S_n$

Then $\lim_{n\to\infty} T_n = \lim_{n\to\infty} k \cdot S_n = k \lim_{n\to\infty} S_n = k \cdot S$

(32) Prove: If $\sum_{n=1}^{\infty} t_k$ and $\sum_{n=1}^{\infty} p_k$ are convergent series with sums T and P respectively, then $\sum_{n=1}^{\infty} (t_k + p_k)$ and $\sum_{n=1}^{\infty} (t_k - p_k)$ are convergent and $T+P$ and $T-P$ are their respective sums.

Let $T_n = t_1 + t_2 + \dots + t_n$ and $P_n = p_1 + p_2 + \dots + p_n$

Then $\lim_{n\to\infty} T_n = T$ and $\lim_{n\to\infty} P_n = P$

Let $R_n = (t_1 + p_1) + (t_2 + p_2) + \dots + (t_n + p_n)$
$= (t_1 + t_2 + \dots + t_n) + (p_1 + p_2 + \dots + p_n) = T_n + P_n$

$\frac{1}{3}$

Then $\lim_{n\to\infty} R_n = \lim_{n\to\infty} (T_n + P_n)$

$= \lim_{n\to\infty} T_n + \lim_{n\to\infty} P_n = T + P$

Let $S_n = (t_1 - p_1) + (t_2 - p_2) + \dots + (t_n - p_n)$

$= (t_1 + t_2 + \dots + t_n) - (p_1 + p_2 + \dots + p_n)$

$= T - P$

Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} (T_n - P_n)$

$= \lim_{n\to\infty} T_n - \lim_{n\to\infty} P_n = T - P$

---

(33) Show that $1 + \frac{1}{2^p} + \frac{1}{4^p} + \dots + \frac{1}{2^{n \cdot p}} + \dots$

is convergent for $p \in \mathbb{N}$, $p > 1$.

$S_1 = 1$, $S_2 = 1 + \frac{1}{2^p} < \frac{3}{2}$

$S_3 = 1 + \frac{1}{2^p} + \frac{1}{4^p} < \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$

$S_n < \frac{2^n - 1}{2^{n-1}}$

$$S_n < \frac{2^n - 1}{2^{n-1}} \qquad \lim_{n\to\infty} S_n < \lim_{n\to\infty} \frac{2^n-1}{2^{n-1}}$$

$$= \lim_{n\to\infty} 2 - \frac{1}{2^{n-1}} = 2$$

The series is nondecreasing and bounded, and thus converges.

(34) Show that the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} \text{ is}$$

convergent. Hint: Write S_n in the form

$$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots$$

$$+ \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

(35) Show that the following series converges and find its sum.

$$\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{9}\right)+\dots+\left(\frac{1}{2^n}+\frac{1}{3^n}\right)+\dots$$

[Hint: Express S_n as the sum of two separate geometric series]

$$S_n = \left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}\right)+\left(\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{3}}{\frac{2}{3}} = 1 + \frac{1}{2} = \frac{3}{2}$$

(36) Given $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ in which all terms are positive and $a_k \geq b_k$ for all k.

Prove that if $\sum_{k=1}^{\infty} b_k$ is divergent, then $\sum_{k=1}^{\infty} a_k$ is divergent.

Since $\sum_{k=1}^{\infty} b_k$ is divergent, its partial sums are unbounded. Since $a_k \geq b_k$ for all k,

the partial sums of $\sum_{k=1}^{\infty} a_k$ are greater than those of $\sum_{k=1}^{\infty} b_k$ and are unbounded. Since an unbounded series cannot converge, it must diverge.

(37) Find the range of values of x for which the series $\sum_{n=1}^{\infty} \left(\frac{2x+1}{2}\right)^n$ is convergent.

$$\left|\frac{2x+1}{2}\right| < 1 \rightarrow -1 < \frac{2x+1}{2} < 1$$

$$-2 < 2x+1 < 2 \rightarrow -3 < 2x < 1$$

$$-\frac{3}{2} < x < \frac{1}{2}$$

(38) If $\frac{p}{q}$ and $\frac{r}{s}$ are any two positive rational numbers such that $\frac{p}{q} < \frac{r}{s}$, show that $\frac{ps+qr}{2sq}$ is a rational number which lies between the given numbers.

$\frac{ps+qr}{2sq}$ is a rational number $(s, q \neq 0)$ because p, q, r, s being integers, which are closed

under addition and multiplication, both the numerator and denominator are integers.

$$\frac{p}{q} < \frac{r}{s}: \quad \frac{ps+qr}{2qs} = \frac{ps}{2qs} + \frac{qr}{2qs}$$

$$= \frac{p}{2q} + \frac{r}{2s} = \frac{1}{2}\frac{p}{q} + \frac{1}{2}\frac{r}{s} < \frac{1}{2}\frac{r}{s} + \frac{1}{2}\frac{r}{s} = \frac{r}{s}.$$

Also $\frac{ps+qr}{2qs} = \frac{1}{2}\frac{p}{q} + \frac{1}{2}\frac{r}{s} > \frac{1}{2}\frac{p}{q} + \frac{1}{2}\frac{p}{q} = \frac{p}{q}$

Thus $\frac{p}{q} < \frac{ps+qr}{2qs} < \frac{r}{s}$

3-9 POSITIVE ROOTS OF POSITIVE NUMBERS

1. Is $\sqrt{a^2} = a$ for all real values of a? Explain. No. When $a < 0$, $\sqrt{a^2} = |a|$

2. For what values of x is $\sqrt{(x-3)^2} = x-3$?
for $x \geq 3$.

3. Express $0{,}0000284$ in the form $p\left(\frac{1}{10}\right)^k$, where $1 \leq p < 10$ and k is an integer.

$0.0000284 = 2.84\left(\frac{1}{10}\right)^5$

(4) Is 1 an element of the solution set of
$\sqrt{2x-1} = x - 2$?

$2x - 1 = (x-2)^2 = x^2 - 4x + 4$

$x^2 - 6x + 5 = (x-1)(x-5) = 0$

$\{x : x = 1 \text{ or } x = 5\} = \{1, 5\}$

What's wrong here?

$\sqrt{2(1)-1} = \sqrt{2-1} = 1$

but $(1) - 2 = -1$ so 1 is not an element of the solution set.

Since we squared both sides without putting any conditions, extraneous solutions may be introduced. Checking the solutions is necessary. While $x = 5$ is a solution, $x = 1$ is not a solution to $\sqrt{2x-1} = x - 2$. It is an extraneous solution introduced because of raising to power 2, an even power.

$x^2 = y^2$ does not imply $x = y$

With any negative power, we lose the sign.

$\{5..10\}$ Given $a = 2+\sqrt{3}$ and $b = 2-\sqrt{3}$. Find each number and state whether it is rational or irrational.

(5) $a+b = (2+\sqrt{3}) + (2-\sqrt{3}) = 4$ (rational)

(6) $a-b = (2+\sqrt{3})-(2-\sqrt{3}) = 2\sqrt{3}$
irrational.

(7) $a \cdot b = (2+\sqrt{3})(2-\sqrt{3}) = 4-3 = 1$
rational

(8) $\sqrt{a} \cdot \sqrt{b} = \sqrt{2+\sqrt{3}} \cdot \sqrt{2-\sqrt{3}}$

$= \sqrt{(2+\sqrt{3})(2-\sqrt{3})} = \sqrt{4-3} = 1$
rational

(9) $a^2 - b^2 = (2+\sqrt{3})^2 - (2-\sqrt{3})^2$
$= (4+4\sqrt{3}+3) - (4-4\sqrt{3}+3)$
$= (7+4\sqrt{3}) - (7-4\sqrt{3})$
$= 8\sqrt{3}$; irrational

(10) $\sqrt{a^2+b^2} = \sqrt{(7+4\sqrt{3})+(7-4\sqrt{3})}$
$= \sqrt{14}$; irrational

Using the approximations $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$, find an approximation for the given number correct to <u>three</u> significant figures.

[Where possible, first express the number in a simpler radical form.]

$$\begin{array}{r} 1.732 \\ \times\ 4 \\ \hline 6.928 \end{array}$$

(11) $2\sqrt{12} = 4\sqrt{3} \approx 4 \cdot (1.732)$

≈ 6.93

(12) $(3\sqrt{2})^3 = 3^3 \cdot (\sqrt{2})^3 = 27 \cdot (2\sqrt{2}) = 54\sqrt{2}$

≈ 76.4

(13) $\dfrac{6}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{6\sqrt{2}}{2} = 3\sqrt{2} \approx 3 \cdot (1.414) \approx 4.24$

(14) $(\sqrt{3} + \sqrt{2})^2 = 3 + \sqrt{6} + \sqrt{6} + 2 = 5 + 2\sqrt{6}$

$= 5 + 2(\sqrt{2})(\sqrt{3}) \approx 5 + 2(1.414)(1.732) \approx 9.898$

≈ 9.90

(15) $5\sqrt{\dfrac{1}{3}} = \dfrac{5}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{5\sqrt{3}}{3} \approx \dfrac{5 \cdot (1.732)}{3} \approx 2.886667$

≈ 2.89

(16) $\sqrt{18} + \sqrt{12} = 3\sqrt{2} + 2\sqrt{3}$

$\approx 3(1.414) + 2(1.732) \approx 7.706$

≈ 7.71

(17) $\dfrac{4}{\sqrt{3}+2} \cdot \dfrac{(2-\sqrt{3})}{(2-\sqrt{3})} = \dfrac{8-4\sqrt{3}}{-3+4} = 8-4\sqrt{3} = 4(2-\sqrt{3})$

$\approx 8 - 4(1.732) \approx 1.072 \approx 1.07$

(18) $\dfrac{\sqrt{3}}{\sqrt{2}-1} \cdot \dfrac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = \dfrac{\sqrt{6}+\sqrt{3}}{2-1} = \sqrt{6}+\sqrt{3}$

$= \sqrt{2}\cdot\sqrt{3} + \sqrt{3} \approx (1.414)\cdot(1.732) + (1.414)$

$\approx 3.863048 \approx 3.86$

{19..22} Name a rational number r and an irrational number s lying between the given numbers.

(19) $1\frac{7}{10}$ and $\sqrt{3} \rightarrow 1.7 < \sqrt{3} < \frac{1732}{1000}$

Therefore a rational number between $\frac{17}{10}$ and $\sqrt{3}$ is $\frac{171}{100}$.

An irrational number between $\frac{17}{10}$ and $\sqrt{3}$ is $\sqrt[3]{5}$

(20) $\frac{3}{4}$ and $\frac{7}{8}$

rational number: $\frac{3}{4} = \frac{12}{16} < \boxed{\frac{13}{16}} < \frac{14}{16} = \frac{7}{8}$

irrational number: $\dfrac{3}{4} = \dfrac{\sqrt{9}}{\sqrt{16}} = \sqrt{\dfrac{9}{16}} < \boxed{} < \sqrt{\dfrac{49}{64}}$

$$\sqrt{\frac{36}{64}} < \boxed{\frac{\sqrt{10}}{4}} < \sqrt{\frac{49}{64}}$$

Is there a more "elegant" or "sophisticated" or "mature" way to approach this?

$r = \frac{4}{5}$, $s = \frac{\sqrt{3}}{2}$

you see, $\frac{3}{4} = 0.75$ and $\frac{7}{8} = 0.875$

$\frac{4}{5} = 0.8$

What about for irrational between 0.75 and 0.875?

We know $\sqrt{3} \approx 1.732$,

Half of $\sqrt{3}$ is $0.5(1.732) \approx 0.866$

Hence $s = \sqrt{3}/2$

(21) $\sqrt{2}$ and $\sqrt{3}$

This is about 1.414 and 1.732 $r = \frac{3}{2} = 1.5$

Think $\sqrt[3]{5} \approx 1.709$

(22) $\frac{1}{2}$ and $\sqrt{0.27} \approx 0.5196$

$r = \frac{51}{100} = 0.51$, $s = \sqrt{0.26} \approx 0.5099$

Using the approximation method, find the first three digits in the decimal representation for each number.

(23) $\sqrt{5} = 2.b_1b_2\ldots$

$(2.2)^2 = 4.84$, $(2.3)^2 = 5.29$ $\therefore b_1 = 2$

$(2.23)^2 = 4.9729$, $(2.24)^2 = 5.0176$ $\therefore b_2 = 3$

Hence $\sqrt{5} \approx 2.23$

(24) $\sqrt[3]{7}$ so $x^3 = 7$

Since $1^3 = 1$ and $2^3 = 8$, a positive number satisfying $x^3 = 7$ would be between 1 and 2.

$\sqrt[3]{7} = 1.b_1b_2\ldots$

$(1.9)^3 \approx 6.859$, $2.0^3 = 8.00$ $\therefore b_1 = 9$

$(1.91)^3 \approx 6.96787$, $(1.92)^3 \approx 7.077$ $\therefore b_2 = 1$

$\sqrt[3]{7} \approx 1.91$

(25) $8 + \sqrt{10}$, $8 + x^2 = 8 + 10$

$x^2 = 10$, Since $3^2 = 9$ and $4^2 = 16$

$3 < x < 4$

$8 + \sqrt{10} = 8 + 3.b_1b_2\ldots$

$(3.1)^2 \approx 9.61$, $(3.2)^2 \approx 10.24$ $\therefore$ $b_1 = 1$

$(3.16)^2 \approx 9.9856$, $(3.17)^2 \approx 10.0489$ $\therefore$ $b_2 = 6$

$8 + \sqrt{10} \approx 8 + 3.16 = 11.16$

(26) $4 - \sqrt[3]{3}$

To find x such that $x^3 = 3$, consider that $1^3 = 1$ and $2^3 = 8$, so $1 < x < 2$, much closer to 1.

$4 - \sqrt[3]{3} = 4 - 1.b_1 b_2 \ldots$

$(1.4)^3 = 2.744$, $(1.5)^3 = 3.375$ $\therefore$ $b_1 = 4$

$(1.44)^3 \approx 2.985984$, $(1.45)^3 \approx 3.048625$ $\therefore$ $b_2 = 4$

$4 - \sqrt[3]{3} \approx 4 - 1.44 = 2.56$

Perform the indicated operation and express the result in simple radical form.

(27) $(2\sqrt{3})(4\sqrt{6}) = 8\sqrt{18} = 8 \cdot 3\sqrt{2} = 24\sqrt{2}$

(28) $(3\sqrt{2} + 1)^2 = 9 \cdot 2 + 6\sqrt{2} + 1 = 19 + 6\sqrt{2}$

(29) $(3\sqrt{2} + 1)(5\sqrt{3} - 1) = 15 \cdot \sqrt{6} - 3\sqrt{2} + 5\sqrt{3} - 1$

(30) $6\sqrt{\frac{1}{3}} - \frac{9}{\sqrt{3}} = \frac{6}{\sqrt{3}} - \frac{9}{\sqrt{3}} = -\frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{3\sqrt{3}}{3} = -\sqrt{3}$

(31) $\frac{5}{2-\sqrt{3}} \cdot \frac{(2+\sqrt{3})}{(2+\sqrt{3})} = \frac{10+5\sqrt{3}}{4-3} = 5(2+\sqrt{3})$

(32) $\frac{7+\sqrt{5}}{3-\sqrt{5}} \cdot \frac{(3+\sqrt{5})}{(3+\sqrt{5})} = \frac{21+7\sqrt{5}+3\sqrt{5}+5}{9-5} = \frac{26+10\sqrt{5}}{4}$

$= \frac{13+5\sqrt{5}}{2}$

Find the positive root(s) of each equation correct to tenths.

(33) $2x^2 = 5 \rightarrow x^2 = \frac{5}{2} = 2.5$

$1^2 = 1$ and $2^2 = 4$ so $1 < x < 2$, closer to 1

$x = 1.b_1 b_2 \ldots$, $(1.5)^2 = 2.25$, $(1.6)^2 = 2.56$

Since 2.56 is closer to 2.5, $\sqrt{2.5} \approx 1.6$

(34) $(2x-1)^2 = 8$

Since $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $2 < 2x-1 < 3$

$3 < 2x < 4$

$\frac{3}{2} < x < 2$

$(2x-1)^2 = 8 \longrightarrow 2x - 1 = \sqrt{8} = 2\sqrt{2}$

$2x = 2\sqrt{2} + 1 \longrightarrow x = \frac{2\sqrt{2}+1}{2}$

$x = \frac{2\sqrt{2}}{2} + \frac{1}{2} = \sqrt{2} + \frac{1}{2} = 1.4 + 0.5 = 1.9$

(35) $(x-2)^2 = \sqrt{2}$

Since $\sqrt{2} \approx 1.414$ and $1^2 = 1$ and $2^2 = 4$,

$1 < x-2 < 2$ $\qquad$ $(x-2)^2 = \sqrt{2}$

$3 < x < 4$ $\qquad$ $x-2 = \pm \sqrt[4]{2}$

$x = 2 \pm \sqrt[4]{2}$

So, for now, let's look at $\sqrt[4]{2}$

We are looking for x such that $x^4 = 2$

$1^4 = 1$, $2^4 = 16$, so $1 < x < 2$

$(1.1)^4 = 1.4641$, $(1.2)^4 = 2.0736$

Since 2.0736 is closer to 2, $\sqrt[4]{2} \approx 1.2$

so $x = 2 \pm 1.2 \rightarrow x = 3.2$ or $x = 0.8$

(36) $(x-1)^3 = 10 \longrightarrow x-1 = \sqrt[3]{10} \rightarrow x = 1 + \sqrt[3]{10}$

Find s such that $s^3 = 10$. $1^3 = 1$, $2^3 = 8$, $3^3 = 27$

so $2 < s < 3$

$s = 2.b_1b_2\ldots$ $\qquad$ $(2.1)^3 = 9.261$, $(2.2)^3 = 10.648$

2.2 is closer... $x = 1 + 2.2 = 3.2$

---

(37) Prove: If r is an irrational number > 0, then $\sqrt[n]{r}$ is an irrational number.

Suppose $\sqrt[n]{r}$ is rational. Then $(\sqrt[n]{r})^n$ is also rational. Since $(\sqrt[n]{r})^n = r$ is an irrational number, this contradicts the hypothesis $\therefore \sqrt[n]{r}$ is an irrational number.

(38) Prove: The reciprocal of an irrational number is an irrational number.

Let r be irrational.

Suppose $\frac{1}{r}$ is rational.

Then $\frac{1}{r} \cdot r$ is irrational.

Since $\frac{1}{r} r = 1$ is rational,

this is a contradiction.

$\therefore \frac{1}{r}$ is irrational

Prove the following properties of radicals for $m, n \in \mathbb{N}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \geq 0$, $b \geq 0$.

(39) $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Show that $(\sqrt[n]{a})^m$ is the nonnegative number whose n^{th} power is a^m. $[(\sqrt[n]{a})^m]^n = (\sqrt[n]{a})^{m \cdot n} = [(\sqrt[n]{a})^n]^m$

$= a^m$.

(40) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Show that $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ is the nonnegative number whose nth power is $\dfrac{a}{b}$.

$$\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n = \frac{\left(\sqrt[n]{a}\right)^n}{\left(\sqrt[n]{b}\right)^n} = \frac{a}{b}$$

(41) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m\cdot n]{a}$

Show that $\sqrt[m]{\sqrt[n]{a}}$ is the nonnegative number whose $m\cdot n^{th}$ power is a.

$$\left(\sqrt[m]{\sqrt[n]{a}}\right)^{m\cdot n} = \left(\left(\sqrt[m]{\sqrt[n]{a}}\right)^m\right)^n = \left(\sqrt[n]{a}\right)^n = a$$

(42) $\sqrt[k\cdot m]{a^{k\cdot n}} = \sqrt[m]{a^n}$

Show that $\sqrt[m]{a^n}$ is the nonnegative number whose $k\cdot m^{th}$ power is $a^{k\cdot n}$.

$$\left(\sqrt[m]{a^n}\right)^{k\cdot m} = \left(\left(\sqrt[m]{a^n}\right)^m\right)^k = \left(a^n\right)^k = a^{k\cdot n}$$

(43) Given the irrational number $\sqrt{a}$, prove that $1+\sqrt{a}$ cannot be rational.

Suppose that $1+\sqrt{a} = r$, a rational number.

Then $\sqrt{a} = r-1$ is also rational.

Contradiction $\therefore$ $1+\sqrt{a}$ is irrational.

(44) Given the irrational numbers $(a+b\sqrt{x})$ and $(c+d\sqrt{x})$ where a, b, c, and d are nonzero rational numbers, determine under what conditions the product of the two numbers is rational.

$$(a+b\sqrt{x})(c+d\sqrt{x}) = ac + a\cdot d\sqrt{x} + cb\sqrt{x} + b\cdot c\cdot x$$

$$= c(a+bx) + \sqrt{x}(ac+bc)$$

Rational if $ac+bc=0$ and x is rational.

(45) Given $a>0$, $b>0$, prove $\dfrac{2ab}{a+b} \leq \sqrt{ab}$

$$(a-b)^2 \geq 0 \rightarrow a^2-2ab+b^2 \geq 0$$

Add $4ab$ to both sides: $a^2+2ab+b^2 \geq 4ab$

$$(a+b)^2 \geq 4ab$$

$$1 \geq \frac{4ab}{(a+b)^2}$$ multiply both sides by $a\cdot b$

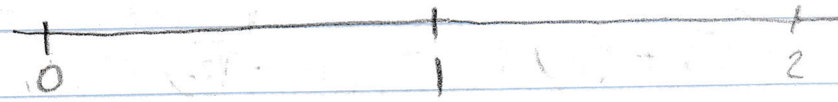
$$ab \geq \frac{4a^2b^2}{(a+b)^2} \rightarrow \sqrt{ab} \geq \frac{2ab}{a+b}$$

(46) Express $\sqrt[3]{2}$ and $\sqrt{3}$ as radicals having the same root index.

$$\sqrt[k\cdot m]{a^{k\cdot n}} = \sqrt[m]{a^n} \quad \text{so} \quad \sqrt[3\cdot 2]{2^2} = \sqrt[6]{4} = \sqrt[3]{2}$$

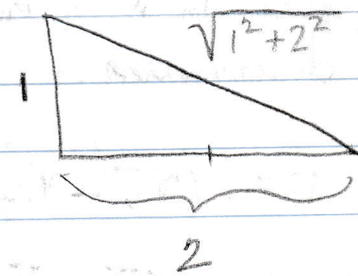
and $$\sqrt[2\cdot 3]{3^3} = \sqrt[6]{27} = \sqrt{3}$$

(47) Given a line segment of unit length, describe a method by which you can construct a segment of length $\sqrt{5}$.



$x^2 = 5$

$2^2 = 4,\ 3^2 = 9,\quad 2 < x < 3$



$\sqrt{1^2+2^2} = \sqrt{5}$

(48) If $b > 0$ and $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{b}} > \sqrt{3} + \sqrt{b}$, find the range of values of b.

$$\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{b}} = \frac{\sqrt{b}+\sqrt{3}}{\sqrt{3b}} > \sqrt{3} + \sqrt{b} \rightarrow \frac{1}{\sqrt{3b}} > 1$$

$$\sqrt{3b} < 1 \rightarrow 0 < 3b < 1 \rightarrow 0 < b < \frac{1}{3}$$

(49) Form an equation with integral coefficients which has $2+\sqrt[3]{3}$ as one of its roots.

For example, let $x = 2+\sqrt[3]{3}$

$x-2 = \sqrt[3]{3} \rightarrow (x-2)^3 = 3$

(50) Find the sum of the infinite geometric series $(\sqrt{3}+\sqrt{2}) + 1 + (\sqrt{3}-\sqrt{2}) + \ldots$ and express your result in a form containing no radicals in the denominator.

$r = (\sqrt{3}-\sqrt{2})$, $a = (\sqrt{3}+\sqrt{2})$

$$S = \frac{\sqrt{3}+\sqrt{2}}{1-(\sqrt{3}-\sqrt{2})} \cdot \frac{(1+(\sqrt{3}-\sqrt{2}))}{(1+(\sqrt{3}-\sqrt{2}))}$$

$$= \frac{\sqrt{3}+\sqrt{3}(\sqrt{3}-\sqrt{2})+\sqrt{2}+\sqrt{2}(\sqrt{3}-\sqrt{2})}{1-(3-2\sqrt{6}+2)}$$

$$= \frac{(\sqrt{3}+\sqrt{2})+(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}{2\sqrt{6}-4}$$

$$= \frac{\sqrt{3}+\sqrt{2}+1}{2(\sqrt{6}-2)} \cdot \frac{\sqrt{6}+2}{\sqrt{6}+2} = \frac{\underbrace{\sqrt{18}}_{3\sqrt{2}}+2\sqrt{3}+\underbrace{\sqrt{12}}_{2\sqrt{3}}+2\sqrt{2}+\sqrt{6}+2}{4}$$

$$= \frac{2+4\sqrt{3}+5\sqrt{2}+\sqrt{6}}{4}$$

Chapter 3 Test

3-1 ① Prove by the method of mathematical induction:

$$1+4+7+\ldots+(3n-2)=\frac{n(3n-1)}{2}$$

Let S be the set of integers $n>0$ for which the statement is true.

(1) $1\in S$: $1=\frac{1(3(1)-1)}{2}=\frac{2}{2}=1$

(2) Assume $x\in S$ and $\sum_{n=1}^{x} 3n-2=\frac{x(3x-1)}{2}$

Then $\sum_{n=1}^{x+1} 3n-2=\frac{(x+1)(3x+2)}{2}$

and $x+1\in S$: $\frac{x(3x-1)}{2}+(3(x+1)-2)$

$$=\frac{3x^2-x}{2}+(3x+1)$$

$$=\frac{3x^2-x+6x+2}{2}=\frac{3x^2+5x+2}{2}=\frac{(x+1)(3x+2)}{2}$$

$a\cdot c=3\cdot2=6=u\cdot v$

$b=5=u+v$

$u=3$

$v=2$

$$3x^2+3x+2x+2=3x(x+1)+2(x+1)$$

$$=(x+1)(3x+2)$$

so $S=\mathbb{N}$

has

your
the

$+\sqrt{6}+2$

3-2 (2) Write the series $\sum_{k=1}^{4} (k^2-k)$ in expanded form

$(1^2-1)+(2^2-2)+(3^2-3)+(4^2-4)$

$= 0+2+6+12 = 20$

3-3 (3) Insert three arithmetic means between 18 and -6.

$a=18$, $n=5$, $t_5=-6$

$-6 = 18+(5-1)d$

$-24 = 4d \rightarrow d=-6$

$\therefore$ 18, $\underline{12}$, $\underline{6}$, $\underline{0}$, -6

(4) Find the sum of the first twenty positive odd integers.

$\sum_{n=1}^{20} (2n-1)$. $a=1$, $d=2$, $n=20$

$t_{20} = 1+(20-1)2$
$= 1+38 = 39$

$1+3+5+\ldots+(2n-1) = \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{20}{2}\left[2\cdot 1+19(2)\right] = 10(2+38) = 400$

3-4 (5) The fifth term of a geometric progression is -2 and the seventh term is -8.
Find the first term.

$t_n = a\cdot r^{n-1}$

$t_5 = -2 = ar^4$ and $t_7 = ar^6 = -8$

so $t_7 = -8 = (-2)r^2 \rightarrow r^2=4$ so $r=2$ or $r=-2$

$-2 = a(2)^4 = 16a$ so $a = \frac{-1}{8}$

⑥ Find the sum of the geometric series

$\sum_{k=1}^{5} 3\left(-\frac{1}{3}\right)^{k-1}$. Solution: $a=3$, $r=-\frac{1}{3}$, $n=5$

$$S_n = \frac{a - a\cdot r^n}{1-r}, \text{ so } S_5 = \frac{3 - 3\cdot\left(-\frac{1}{3}\right)^5}{1-\left(-\frac{1}{3}\right)}$$

$$S_5 = \frac{3 - 3\left(-\frac{1}{243}\right)}{1+\frac{1}{3}} = \frac{3+\frac{1}{81}}{\frac{4}{3}} = \frac{\frac{243+1}{81}}{\frac{4}{3}}$$

$$= \frac{244}{81}\cdot\frac{3}{4} = \frac{244}{27\cdot 4} = \frac{244}{108} = \frac{122}{54} = \frac{61}{27}$$

3-5 ⑦ Write the first three terms in the expansion of $(2x-3y)^5$.

$$(2x-3y)^5 = (2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2$$

$$= 32x^5 - 240x^4y + 720x^3y^2 + \ldots$$

3-6 ⑧ Given $a_n = \frac{n(n-2)}{3n^2+2n}$, find $\lim_{n\to\infty} a_n$.

$$a_n = \frac{n^2-2n}{3n^2+2n} = \frac{1-\frac{2}{n}}{3+\frac{2}{n}}; \quad \lim_{n\to\infty} a_n = \frac{1}{3}$$

3-7 ⑨ Write the decimal equivalent of $\frac{2}{7}$.

$$\begin{array}{r} 0.\overline{285714}28 \\ 7 \,)\, 2.00000000 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \end{array}$$

$$\frac{2}{7} = 0.\overline{285714}$$

(10) Find the sum of the infinite geometric progression:

$4, -2, 1, \ldots \qquad a = 4$

$r = -\frac{2}{4} = -\frac{1}{2}$

$$S = \frac{a}{1-r} = \frac{4}{1-(-\frac{1}{2})} = \frac{4}{1+\frac{1}{2}} = \frac{4}{3/2}$$

$$= 4 \cdot \frac{2}{3} = \frac{8}{3}$$

3-8 (11) Find the first three terms of the sequence of partial sums of the series

$$\frac{1}{3} + \frac{1}{9} + \ldots + \left(\frac{1}{3}\right)^n$$

$S_1 = \frac{1}{3}, \quad S_2 = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

$S_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{4}{9} + \frac{1}{27} = \frac{12}{27} + \frac{1}{27} = \frac{13}{27}$

(12) Write a convergent geometric series whose sum bounds the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{5^n+1}\right).$$

$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$; since $5^n+1 > 5^n$, $\frac{1}{5^n+1} < \frac{1}{5^n}$

ressions: 3-9 13 Name an irrational number lying between $\sqrt{3}$ and $\sqrt{5}$. $\sqrt{4}$

14 Find the positive root of $x^3 = 11$ correct to hundredths without using tables.

$2^3 = 8$, $3^3 = 27$ $\therefore$ $2 < x < 3$

$x = 2.b_1 b_2 \ldots$

$(2.2)^3 \approx 10.648$, $(2.3)^3 \approx 12.167$ $\therefore$ $b_1 = 2$

$(2.22)^3 \approx 10.94105$, $(2.23)^3 \approx 11.08957$ $\therefore$ $b_2 = 2$

Hence $x = 2.22$ (to be more exact, $x \approx 2.224$)

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1 The Algebra of Vectors

4-1

1 If $K = \{a\}$ and $M = \{b\}$, specify $K \times M$ and $M \times K$ by roster.

$K \times M = \{(a, b)\}$

$M \times K = \{(b, a)\}$

2 If $K \times M = \{(a, b), (a, c)\}$, specify K and M by roster.

$K = \{a\}$, $M = \{b, c\}$

n

(3) If K is not the empty set, is K a subset of $K \times M$?

No, $K \times M$ is a set of ordered pairs.

(4) If K is the empty set and M a nonempty set, is $K \times M$ a set of ordered pairs? Why or why not?

No. $K \times M = \emptyset$ since there are no elements in K for abscissas.

(5) Find x and y if $(x+3, y-2)$ and $(6, 4)$ have the same point as graph.

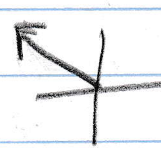
$x + 3 = 6 \iff x = 3$

$y - 2 = 4 \iff y = 6$

(6) Name the quadrant in which the graph of (a,b) lies if $b > 0$ and $ab < 0$.

This means $a < 0$.

$\therefore$ (a,b) is in the second quadrant.



(7) Is $(-3, 2) \in \{(x,y) : 2x - 3y = 12\}$?

$2(-3) - 3(2) = -6 - 6 = -12 \neq 12$

$\therefore$ NO

(8) If $(2,-1) \in \{(x,y) : 2x + ky = 16\}$, determine k.

$2(2) + k(-1) = 16 \leftrightarrow 4 - 16 = k \leftrightarrow k = -12$

(9) Is the solution set of $\{(x,y) : 2x + 3y = 18\}$ a finite set? Why or why not? No.

Why not? For any $x_1 \in \mathbb{R}$, there is a unique $y_1 \in \mathbb{R}$ such that $2x_1 + 3y_1 = 18$;

given an $x_1 \in \mathbb{R}$, $2x_1 + 3y_1 = 18$, $3y_1 = 18 - 2x_1$,

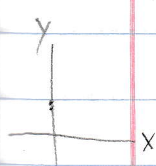
$y_1 = \dfrac{18 - 2x_1}{3} \in \mathbb{R}$.

If $x_2 \neq x_1$, $y_2 = \dfrac{18 - 2x_2}{3} \neq \dfrac{18 - 2x_1}{3} = y_1$,

and so $y_2 \neq y_1$ $\therefore$ the solution set is infinite.

(10) If $K = \{0\}$ and $S = \{0\}$, name the point that is the graph of $K \times S$. $(0,0)$

(11) If the graph of (a,b) is a point on the y-axis, which coordinate of (a,b) is zero?



$a = 0$ (abscissa). $(a,b) \in \{(x,y) : x = 0, y \in \mathbb{R}\}$

$= \{(0,y) : y \in \mathbb{R}\} =$ y-axis

(12) Given: $I = \{\text{positive integers}\}$; $K = \{\text{negative integers}\}$

In which quadrant will the graph of each element of $I \times K$ lie? of $K \times I$? $I \times K$: IV; $K \times I$: II

(13) Given: $X = \{\text{real numbers}\}$;
$Y = \{\text{irrational numbers}\}$.

Will there be a one-to-one correspondence between the elements of $X \times Y$ and all the points in a coordinate plane? EXPLAIN.

There is a one-to-one correspondence if every value for the ordered pairs of $X \times Y$ is associated with a unique value in the coordinate plane, making it possible to find an "inverse function":

$\mathbb{R} \times \mathbb{R}$

$\{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ = coordinate plane

$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$ is the set of all points with an irrational number for its second entry.

There is a one-to-one correspondence between $X \times Y$ and the subset of $\mathbb{R} \times \mathbb{R}$ that has an irrational number for its second entry.

Thus a point with a rational number for its second entry would have no

corresponding element in $X \times Y$ so that there is not a one-to-one correspondence between $X \times Y$ and the coordinate plane $\mathbb{R} \times \mathbb{R}$.

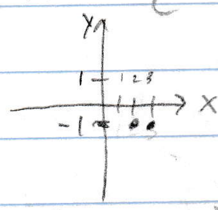
{14., 19} Specify by roster (a) the set $X \times Y$, (b) the set $Y \times X$
(c) Represent the elements $X \times Y$ as points in a coordinate plane.

(14) $X = \{3, 2\}$, $Y = \{-1\}$

(a) $X \times Y = \{(3,-1), (2,-1)\}$

(b) $Y \times X = \{(-1,3), (-1,2)\}$

(c)

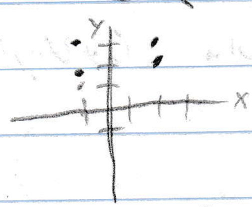


(15) $X = \{2, -1\}$, $Y = \{2, 3\}$

(a) $X \times Y = \{(2,2), (2,3), (-1,2), (-1,3)\}$

(b) $Y \times X = \{(2,2), (2,-1), (3,2), (3,-1)\}$

(c)

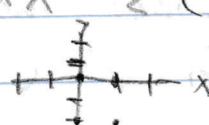


(16) $X = \{0, 1\}$, $Y = \{0, -2\}$

(a) $X \times Y = \{(0,0), (0,-2), (1,0), (1,-2)\}$

(b) $Y \times X = \{(0,0), (-2,0), (0,1), (-2,1)\}$

(c)

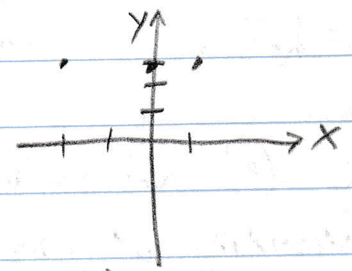


(17) $X = \{-2, 0, 1\}$, $Y = \{3\}$

(a) $X \times Y = \{(-2,3), (0,3), (1,3)\}$

(b) $Y \times X = \{(3,-2), (3,0), (3,1)\}$

(c)



(18) TOO EASY! Move along!

On the basis of the definition of equality of ordered pairs, show that the following equality properties hold for ordered pairs of real numbers.

(20) $(a,b) = (a,b)$ (Reflexive)

By the definition of equality for ordered pairs, $a = a$ and $b = b$, $\therefore (a,b) = (a,b)$

(21) If $(a,b) = (c,d)$, then $(c,d) = (a,b)$ (Symmetric)

Given $(a,b) = (c,d)$ $\therefore$ $a = c$ and $b = d$ by def of equality for ordered pairs.

By symmetry for equality of real numbers $c = a$ and $d = b$ $\therefore (c,d) = (a,b)$.

(22) If $(a, b) = (c, d)$ and $(c, d) = (r, s)$,
then $(a, b) = (r, s)$ [TRANSITIVE]

Since $(a, b) = (c, d)$, by definition of equality for ordered pairs, $a = c$ and $b = d$.

Similarly, since $(c, d) = (r, s)$, $c = r$ and $d = s$.
By transitivity for equality of real numbers,
$a = c = r$ and $b = d = s$
$\therefore$ $a = r$ and $b = s$

$\therefore$ by definition of equality of ordered pairs,
$(a, b) = (r, s)$.

{23, 30} Find the values of x and y for which each of the following is true.

(23) $(3x-2, y+1) = (7, 4)$ $3x-2=7$, $y+1=4$
$3x = 9$ $y = 3$
$x = 3$

(24) $(x^2, 2y-1) = (9, 3)$ $x = 3$ or $x = -3$
$2y-1 = 3 \Leftrightarrow 2y = 4$
$y = 2$

(25) $(x-1, \sqrt{y}) = (-4, 3)$ $x-1=-4 \leftrightarrow x = -3$
$\sqrt{y} = 3 \leftrightarrow y = 9$

(26) $(|x|, |y|) = (3,5)$

$x = 3$ or $x = -3$
$y = 5$ or $y = -5$

(27) $(2x-1, x+2) = (3,4)$

$2x-1 = 3 \Leftrightarrow x = 2$
$x+2 = 4 \Leftrightarrow x = 2$

(28) $(x^2+3x, 2x+1) = (0,7)$

$x(x+3) = 0$	$2x = 6$
$x = -3$	$x = 3$
or $x = 0$	

There are no values of x such that $x^2+3x = 0$ AND $2x+1 = 7$

(29) $(x^2+2x-3, 2x+7) = (0,1)$

$x^2+2x-3 = 0$	$2x+7 = 1$	$\therefore$ $x = -3$
$(x+3)(x-1) = 0$	$2x = -6$	
$x = -3$ or $x = 1$	$x = -3$	

(30) $(x^2+x-4, 3x-2) = (2,4)$

$x^2+x-4 = 2$	$3x-2 = 4$	$\therefore$ $x = 2$
$x^2+x-6 = 0$	$3x = 6$	
$(x+3)(x-2) = 0$	$x = 2$	
$x = -3$ or $x = 2$		

(31) Under what circumstances is $A \times B = B \times A$ a true statement?

When the elements of A are the same as the elements of B.

Now, what would a "formal" answer look like?

$A = B$;

$A \times B = \{ (x,y) : x \in A \text{ and } y \in B \}$

$= B \times A = \{ (y,x) ; y \in B \text{ and } x \in A \}$

when sets have the same ordered pairs for elements.
This occurs when for any $x \in A$, $y \in B$, $(x,y) \in A \times B$ and $c \in B$, $d \in A$, $(c,d) \in B \times A$ are such that $x = c$ and $y = d$.

For this to be true for all (x,y) and (c,d), A must equal B.

(32) A is a finite set containing n elements and B a finite set containing s elements. How many elements are contained in $A \times B$? in $B \times A$?
Each has $n \cdot s$ elements.

Formally, for each of the n elements in A, (chosen as abscissa in $A \times B$ and ordinate in $B \times A$) the s elements of B would yield s ordered pairs.
$\therefore$ there would be $n \cdot s$ possible ordered pairs.

(33) $M \times N$ is a finite set containing $(n+1)!$ ordered pairs. If M contains $n!$ elements, how many elements are contained in N?

N has $(n+1)$ elements since $(n+1) \cdot n! = (n+1)!$

Since number of elements in N is $\dfrac{M \times N}{N}$,

$$N = \frac{(n+1)!}{n!} = n+1$$

(34) The ordered pairs $(3,2)$, $(4,1)$, and $(5,3)$ are among the elements in the set $X \times Y$. Name six other ordered pairs belonging to $X \times Y$.

$(3,1)$, $(3,3)$, $(4,2)$, $(4,3)$, $(5,2)$, $(5,1)$

Specify K by roster, given that

(35) $K = \{(x,y) : (x+y,\ 2x-y) = (6,3)\}$

$$x + y = 6 \quad \longleftrightarrow \quad -2x - 2y = -12$$

$$2x - y = 3 \qquad\qquad 2x - y = 3$$

$$-3y = -9$$

$$x + 3 = 6$$

$$x = 3 \qquad\qquad y = \frac{-9}{-3} = 3$$

$\therefore \; K = \{(3,3)\}$

(36) $K = \{(x,y) : (2x+y,\ x+y) = (4,6)\}$

$$\begin{matrix} 2x+y=4 \\ x+y=6 \end{matrix} \longleftrightarrow \begin{matrix} 2x+y=4 \\ -2x-2y=-12 \\ \hline -y=-8 \leftrightarrow y=8 \end{matrix}$$

$x+8=6 \leftrightarrow x=-2$

$K = \{(-2,8)\}$

(37) $K = \{(x,y) : (y-2,\ 2x-1) = (x,y)\}$

$$\begin{matrix} y-2=x \\ 2x-1=y \end{matrix} \longleftrightarrow \begin{matrix} 2(y-2)-1=y \\ 2y-4-1=y \\ y=5 \end{matrix}$$

$5-2=3=x \quad \therefore \quad K = \{(3,5)\}$

(38) $K = \{(x,y) : (y-x^2,\ y-x) = (0,0)\}$

$$\begin{matrix} y-x^2=0 \\ y-x=0 \end{matrix} \longleftrightarrow x=y \longleftrightarrow y-y^2=0$$

$y(1-y)=0 \qquad y=0$ or $y=1$

so $x=0$ AND $y=0$ OR $x=1$ AND $y=1$

~~How would we write this~~ "formally"?

$K = \{(0,0),(1,1)\}$

(39) $K = \{(x,y) : (|x+1|, |y+1|) = (2,3)\}$

$|x+1| = 2$ and $|y+1| = 3$

$x+1 = 2$ or $x+1 = -2$ | $y+1 = 3$ OR $y+1 = -3$

$x = 1$ $x = -3$ | $y = 2$ $y = -4$

$K = \{(1,2), (1,-4), (-3,2), (-3,-4)\}$

(40) If $R = \{1,2,4\}$ and $S = \{2,4,5\}$, specify the set $(R \cap S) \times (R \cup S)$ by roster.

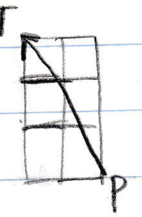
$(R \cap S) = \{2,4\}$

$(R \cup S) = \{1,2,4,5\}$

$(R \cap S) \times (R \cup S) = \{(2,1), (2,2), (2,4), (2,5), (4,1), (4,2), (4,4), (4,5)\}$

$\boxed{4\text{-}2}$ $\{1,,6\}$ Can be done in head.

$\{7,,12\}$ Let $\overrightarrow{PT}$ be a representation of the ordered pair $(-2,3)$. Find the coordinates of T when the coordinates of P are



(7) $P(0,0)$:
$T(0-2, 0+3) = (-2,3)$

(8) $P(5,2)$ so $T(5-2, 2+3) = (3,5)$

(9) $P(2,-1)$ so $T(2-2, -1+3) = (0,2)$

(10) $P(-3,4)$ so $T(-3-2, 4+3) = (-5,7)$

(11) $P(-2,-3)$ so $T(-2-2, -3+3) = (-4,0)$

(12) $P(a,b)$ so $T(a-2, b+3) = (a-2, b+3)$

$\{13,,20\}$ Name the ordered pair which $\overrightarrow{RS}$ represents when the respective coordinates of R and S are

(13) $(3,2)$; $(4,6)$
$\overrightarrow{RS}$ represents $(4-3, 6-2) = (1,4)$

(14) $(-2,1)$; $(3,-5)$
$\overrightarrow{RS}$ represents $(3-(-2), -5-1) = (5,-6)$

(15) $(0,0)$; $(3,-2)$
$\overrightarrow{RS}$ represents $(3-0, -2-0) = (3,-2)$

(16) $(0,3)$; $(5,-2)$
$\overrightarrow{RS}$ represents $(5-0, -2-3) = (5,-5)$

(17) $(2,3)$; $(2,3)$
$\overrightarrow{RS}$ represents $(0,0)$

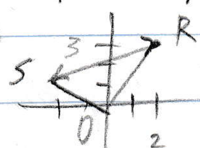
(18) $(a,b);(2a,3b)$; $\overrightarrow{RS}$ represents $(2a-a,\ 3b-b)$
$= (a, 2b)$

(19) $(a,b);(c,d)$ $\overrightarrow{RS}$ represents $(c-a,\ d-b)$

(20) $(a+k,\ b+t);\ (2a-k,\ 2b+3t)$
$\overrightarrow{RS}$ represents $(2a-k-a-k,\ 2b+3t-b-t)$
$= (a-2k,\ b+2t)$

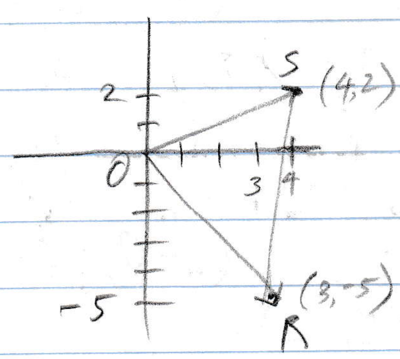
$\{21, 26\}$
Let $\overrightarrow{OR}$ and $\overrightarrow{OS}$ be the respective representations of the ordered pair named. In each exercise:
(a) sketch $\overrightarrow{OR}$, $\overrightarrow{OS}$, and $\overrightarrow{RS}$; and
(b) name the ordered pair represented by $\overrightarrow{SR}$.

(21) $\overrightarrow{OR}$ $(2,3)$; $\overrightarrow{OS}$ $(-2,1)$ (a)



$\overrightarrow{SR} = (2-(-2),\ 3-1) = (4,2)$

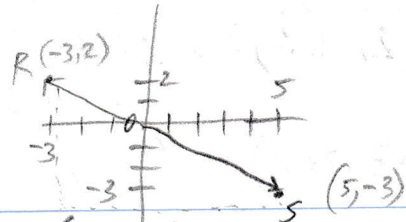
(22) $\overrightarrow{OR}$ $(3,-5)$; $\overrightarrow{OS}$ $(4,2)$



$\overrightarrow{SR} = (3-4,\ -5-2)$
$= (-1,-7)$

(23) $\overrightarrow{OR}$ $\overrightarrow{OS}$

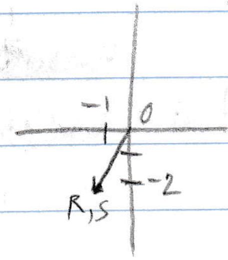
$(-3,2);\ (5,-3)$



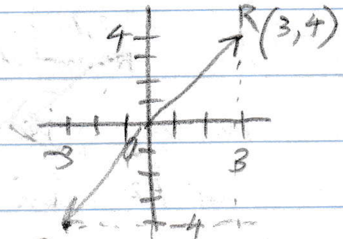
$\overrightarrow{SR} = (-3-5,\ 2-(-3)) = (-8,5)$

(24) $(-1,-2);\ (-1,-2)$

$\overrightarrow{SR} = (0,0)$



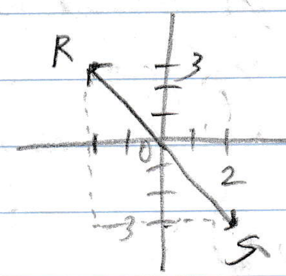
(25) $(3,4);\ (-3,-4)$



$\overrightarrow{SR} = (3-(-3),\ 4-(-4))$
$= (6,8)$

(26) $\overrightarrow{OR}$ $\overrightarrow{OS}$

$(-2,3);\ (2,-3)$

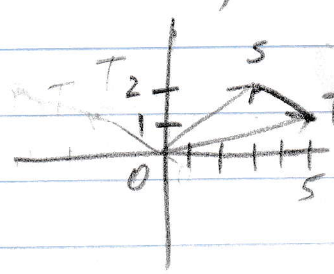


$\overrightarrow{SR} = (-2-2,\ 3-(-3))$
$= (-4,6)$

{27,..32} Let $\overrightarrow{OT}$ be the standard representation of the first ordered pair named. Let $\overrightarrow{TS}$ be a representation of the second ordered pair. In each exercise (a) name the ordered pair represented by $\overrightarrow{OS}$ (b) sketch $\overrightarrow{OT}$, $\overrightarrow{TS}$, and $\overrightarrow{OS}$

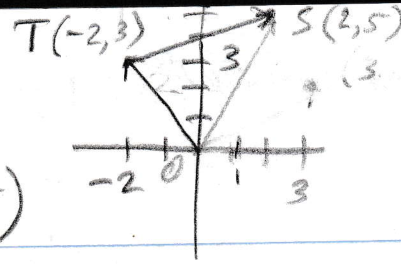
(27) $\overrightarrow{OT}$ $\overrightarrow{TS}$

$(5,1);\ (-2,1)$



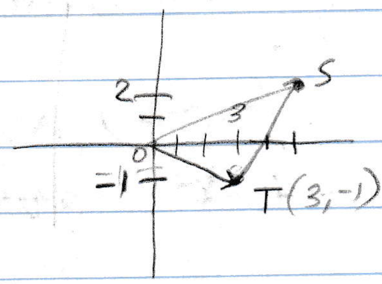
$\overrightarrow{OS} = (5-2,\ 1+1) = (3,2)$

(28) $\overset{OT}{(-2,3)}; \overset{TS}{(3,2)}$



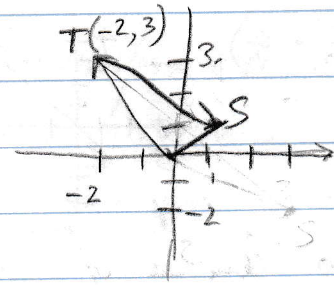
$\vec{OS} = (-2+3, 3+2) = (1,5)$

(29) $\overset{\vec{OT}}{(3,-1)}; \overset{\vec{TS}}{(2,3)}$



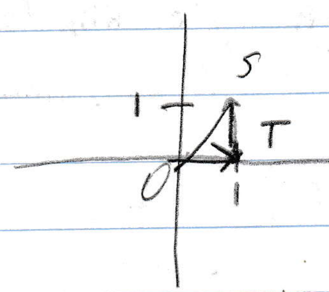
$\vec{OS} = (3+2, -1+3)$
$= (5, 2)$

(30) $\overset{\vec{OT}}{(-2,3)}; \overset{\vec{TS}}{(3,-2)}$



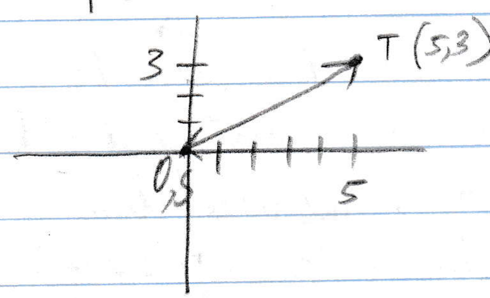
$\vec{OS} = (-2+3, 3-2)$
$= (1, 1)$

(31) $\overset{\vec{OT}}{(1,0)}; \overset{\vec{TS}}{(0,1)}$



$\vec{OS} = (1, 1)$

(32) $\overset{\vec{OT}}{(5,3)}; \overset{\vec{TS}}{(-5,-3)}$



$\vec{OS} = (5-5, 3-3) = (0,0)$

(33) Given three points R, S, and T with respective coordinates (a_1, b_1), (a_2, b_2), and (a_3, b_3). Find the coordinates of point X such that

(a) $\overrightarrow{RS}$ and $\overrightarrow{TX}$ represent the same ordered pair.
(b) $\overrightarrow{RS}$ and $\overrightarrow{XT}$ represent the same ordered pair.
(c) $\overrightarrow{RX}$ and $\overrightarrow{TS}$ represent the same ordered pair.

(a) The vector $\overrightarrow{RS}$ represents the displacement from point $R(a_1, b_1)$ to point $S\ (a_2, b_2)$. Hence $\overrightarrow{RS}$ represents $(a_2 - a_1, b_2 - b_1)$.

Let the coordinates of X be (x_1, x_2)

$\overrightarrow{TX}$ represents the displacement from point $T(a_3, b_3)$ to point $X\ (x_1, x_2)$. Hence $\overrightarrow{TX}$ represents $(x_1 - a_3, x_2 - b_3)$

So, for $\overrightarrow{RS} = \overrightarrow{TX}$, $x_1 - a_3 = a_2 - a_1$ and $x_2 - b_3 = b_2 - b_1$

$x_1 = a_2 - a_1 + a_3$ and $x_2 = b_2 - b_1 + b_3$

$X: (a_3 + a_2 - a_1, b_3 + b_2 - b_1)$

(b) $\overrightarrow{XT}: (a_3 - x_1, b_3 - x_2)$ | $a_3 - x_1 = a_2 - a_1$

$\overrightarrow{RS}: (a_2 - a_1, b_2 - b_1)$ | $b_3 - x_2 = b_2 - b_1$

$x_1 = a_3 - a_2 + a_1$, $x_2 = b_3 - b_2 + b_1$

$X: (a_3 - a_2 + a_1, b_3 - b_2 + b_1)$

(c) $\overrightarrow{RX} : (x_1 - a_1, x_2 - b_1)$

$\overrightarrow{TS} : (a_2 - a_3, b_2 - b_3)$

so $x_1 - a_1 = a_2 - a_3 \Leftrightarrow x_1 = a_2 - a_3 + a_1$

$x_2 - b_1 = b_2 - b_3 \Leftrightarrow x_2 = b_2 - b_3 + b_1$

$X : (a_2 - a_3 + a_1, b_2 - b_3 + b_1)$

(34) Given the three points A, B, C in a coordinate plane, Prove that there is one and only one point D such that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same ordered pair.

Let $A : (a_1, a_2)$, $B : (b_1, b_2)$, $C : (c_1, c_2)$, $D : (d_1, d_2)$.

$\overrightarrow{AB} : (b_1 - a_1, b_2 - a_2)$, $\overrightarrow{CD} : (d_1 - c_1, d_2 - c_2)$

$d_1 - c_1 = b_1 - a_1 \qquad d_2 - c_2 = b_2 - a_2$

$d_1 = c_1 + b_1 - a_1 \qquad d_2 = c_2 + b_2 - a_2$

Suppose there is a second such point D' with coordinates (d_3, d_4). Then $(b_1 - a_1, b_2 - a_2) = (d_3 - c_1, d_4 - c_2)$

so $d_3 - c_1 = b_1 - a_1$

$d_3 = c_1 + b_1 - a_1 \quad \therefore d_1 = d_3$;

$d_4 - c_2 = b_2 - a_2$

$d_4 = c_2 + b_2 - a_2$

$(d_1$

$\therefore \; (d_1, d_2) = (d_3, d_4)$

$\boxed{4\text{-}3}$ $\{5,..8\}$ Give the additive inverse of each of the following vectors.

(5) $(3, 2) + (-3, -2) = \vec{0}$

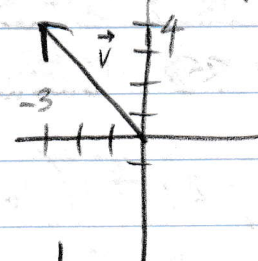
(6) $(-1, 3) + (1, -3) = \vec{0}$

(7) $(4, 3-\sqrt{2}) + (-4, \sqrt{2}-3) = \vec{0}$

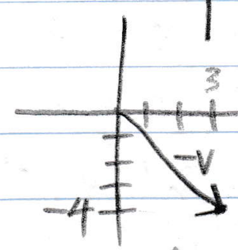
(8) $(-3, -1+\sqrt{5}) + (3, 1-\sqrt{5}) = \vec{0}$

$\{9,..12\}$ Given $\vec{v} = (-3, 4)$, represent by an arrow in standard position

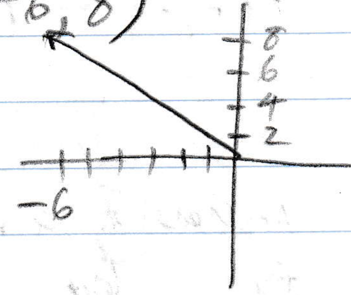
(9) $\vec{v} = (-3, 4)$



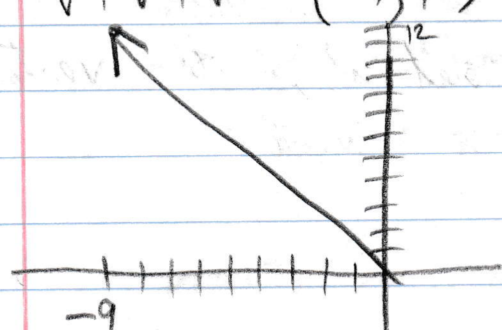
(10) $-\vec{v} = (3, -4)$



(11) $\vec{v} + \vec{v} = (-3 + (-3), 4 + 4) = (-6, 8)$



(12) $\vec{v} + \vec{v} + \vec{v} = (-9, 12)$



{17..22} Find r and s so that each statement is true.

(17) $(r,s) + (3,2) = (5,7)$

$(r,s) = (5-3,\ 7-2) = (2,5)$

(18) $(5,-2) + (r,s) = (7,3)$

$(r,s) = (7-5,\ 3-(-2)) = (2,5)$

(19) $(2,3) + (r,s) = (2,3)$

$(r,s) = (2-2,\ 3-3) = (0,0)$

(20) $(5,r) + (s,3) = (8,-1)$

$5+s=8 \Leftrightarrow s=3$

$r+3=-1 \Leftrightarrow r=-4$

(21) $(-3,r) + (s,-2) = (0,0)$

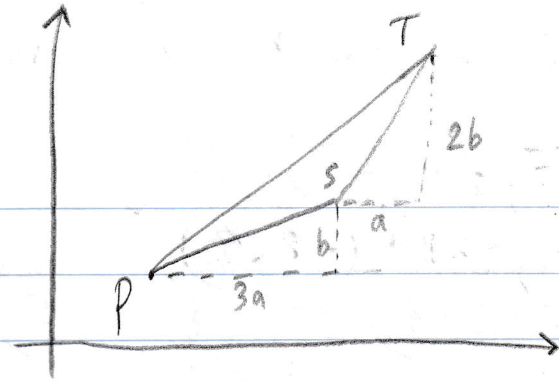
$-3+s=0 \Leftrightarrow s=3$

$r-2=0 \Leftrightarrow r=2$

(22) $(4,-1) + (3,1) = (r,s)$

$(r,s) = (4+3,\ -1+1) = (7,0)$

{23..28} From the coordinate plane vector representation shown, determine the vector represented by the arrow named



(23) $\overrightarrow{PT}$: $(4a, 3b)$

(24) $\overrightarrow{PS}$: $(3a, b)$

(25) $\overrightarrow{ST}$: $(a, 2b)$

(26) $\overrightarrow{TS}$: $(-a, -2b)$ (27) $\overrightarrow{SP}$: $(-3a, -b)$

(28) $\overrightarrow{TP}$: $(-4a, -3b)$

{29, .34} If $\vec{s} = (3,2)$, $\vec{t} = (-4,3)$, and $\vec{u} = (2,-4)$, find $\vec{v}$ satisfying each equation.

(29) $\vec{v} = \vec{s} + \vec{t} = (3,2) + (-4,3) = (3-4, 2+3) = (-1,5)$

(30) $\vec{v} = \vec{t} - \vec{u} = \vec{t} + (-\vec{u}) = (-4,3) + (-2,4)$
$= (-4-2, 3+4) = (-6,7)$

(31) $\vec{t} = \vec{s} + \vec{v} \longleftrightarrow \vec{v} = \vec{t} - \vec{s} = \vec{t} + (-\vec{s})$
$= (-4,3) + (-3,-2) = (-4-3, 3-2) = (-7,1)$

(32) $\vec{v} - \vec{s} = \vec{u} + \vec{t} \longleftrightarrow \vec{v} = \vec{s} + \vec{u} + \vec{t}$
$= (3,2) + (2,-4) + (-4,3)$
$= (3+2-4) + (2-4+3) = (1,1)$

(33) $\vec{v} = \vec{s} - \vec{t} + \vec{u} = \vec{s} + (-\vec{t}) + \vec{u}$
$= (3,2) + (4,-3) + (2,-4) = (3+4+2, 2-3-4) = (9,-5)$

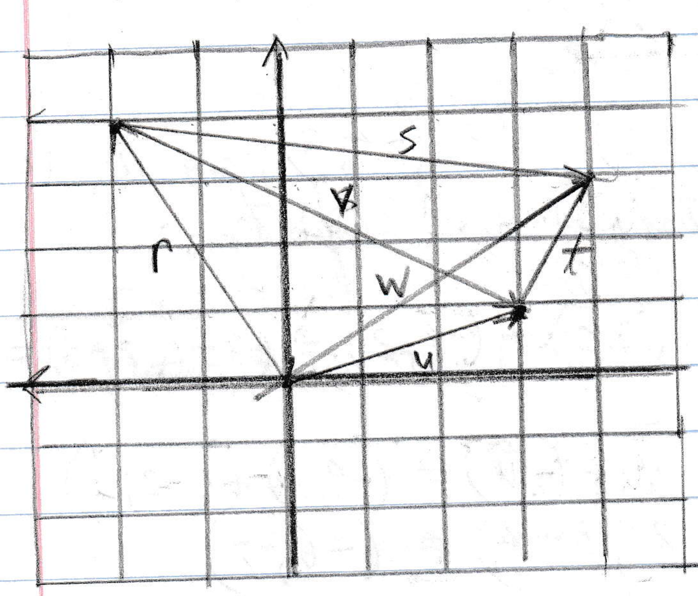
(34) $\vec{t} = \vec{s} + \vec{u} + \vec{v}$

$\vec{v} = \vec{t} - \vec{s} - \vec{u} = \vec{t} + (-\vec{s}) + (-\vec{u})$

$= (-4,3) + (-3,-2) + (-2,4)$

$= (-4-3-2,\ 3-2+4) = (-9,5)$

(35) From the given figure, determine vectors $\vec{r}$, $\vec{s}$, $\vec{t}$, $\vec{u}$, $\vec{v}$, $\vec{w}$.



$\vec{r} = (-2,4)$

$\vec{s} = (6,-1)$

$\vec{t} = (1,2)$

$\vec{u} = (3,1)$

$\vec{v} = (5,-3)$

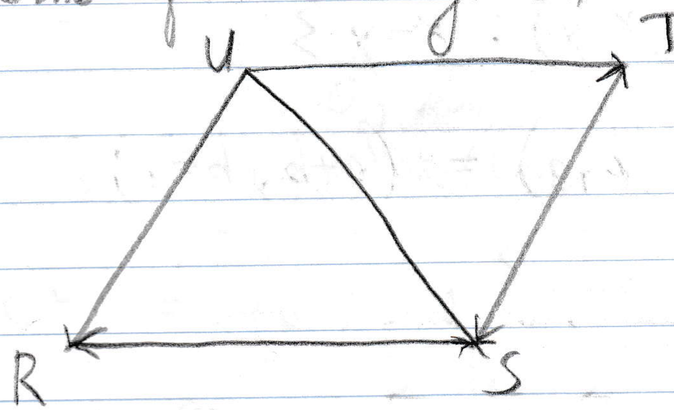
$\vec{w} = (-4,-3)$

(36) Express the vector $(3,-2)$ as a sum using only the vectors $(1,0)$, $(0,1)$ and their additive inverse.

$(3,-2) = (1,0) + (1,0) + (1,0) - (0,1) - (0,1)$

$= (1,0) + (1,0) + (1,0) + (0,-1) + (0,-1)$

Given the parallelogram RSTU. Sketch the arrows connecting the given vertices and express $\overrightarrow{US}$ in terms of the designated vectors.



(37) $\overrightarrow{UT}$ and $\overrightarrow{TS}$. $\overrightarrow{US} = \overrightarrow{UT} + \overrightarrow{US}$

(38) $\overrightarrow{UT}$ and $\overrightarrow{UR}$. $\overrightarrow{US} = \overrightarrow{UT} + \overrightarrow{UR}$ $(\overrightarrow{UR} = \overrightarrow{TS})$

(39) $\overrightarrow{RS}$ and $\overrightarrow{ST}$. $\overrightarrow{US} = \overrightarrow{RS} - \overrightarrow{ST}$ $(-\overrightarrow{ST} = \overrightarrow{TS} = \overrightarrow{UR})$

(40) $\overrightarrow{SR}$ and $\overrightarrow{ST}$. $\overrightarrow{US} = -\overrightarrow{SR} - \overrightarrow{ST}$

$(-\overrightarrow{SR} = \overrightarrow{RS} = \overrightarrow{UT},\ -\overrightarrow{ST} = \overrightarrow{TS}$

(41) Given: $\overrightarrow{AB}$ having (a, b) as initial-point coordinates and (c, d) as terminal-point coordinates. Prove that $\overrightarrow{AB} + \overrightarrow{BA} = \vec{O}$

$\overrightarrow{AB}: (c-a, d-b)$ and $\overrightarrow{BA}: (a-c, b-d)$

$$\overrightarrow{AB} + \overrightarrow{BA} = (c-a, d-b) + (a-c, b-d)$$

$$= (c-a+a-c,\ d-b+b-d) = \vec{O}$$

(42) Given $\vec{v_1} = (a, b)$ and $\vec{v_2} = (b, a)$.

If $\vec{v_3} = \vec{v_1} + \vec{v_2}$ prove that

$\vec{v_3} \in \{(x, y) : x = y\}$

$\vec{v_3} = (a, b) + (b, a) = (a+b, b+a)$

$\vec{v_3} \in \{(x, y) ; x = y\}$ since $a+b = b+a$

{43,, 46} Let $\vec{s}, \vec{t}, \vec{u}$, and $\vec{v}$ be respectively the vectors (a, b), (c, d), (e, f), and (g, h). Using the definition of vector addition, prove each statement.

(43) $\vec{s} + \vec{t} = \vec{t} + \vec{s}$ (commutative property of vector addition)

$(a, b) + (c, d) = (c, d) + (a, b)$

$(a+c, b+d) = (c+a, d+b)$

(44) $(\vec{s} + \vec{t}) + \vec{u} = \vec{s} + (\vec{t} + \vec{u})$

(associative property of vector addition)

$(\vec{s} + \vec{t}) + \vec{u} = [(a,b) + (c,d)] + (e, f)$

$= (a+c, b+d) + (e, f) = (a+c+e, b+d+f)$

$\vec{s} + (\vec{t} + \vec{u}) = (a,b) + [(c, d) + (e, f)] = (a,b) + (c+e, d+f)$

$= (a+c+e, b+d+f)$

(45) $\vec{v} + (\vec{u} - \vec{v}) = \vec{u}$

$(g,h) + [(e,f) - (g-h)] = (g,h) + (e-g, f-h)$

$= (e,f)$

(46) If $\vec{u} + \vec{v} = \vec{0}$, then $\vec{u} = -\vec{v}$

$(e,f) + (g,h) = (0,0)$

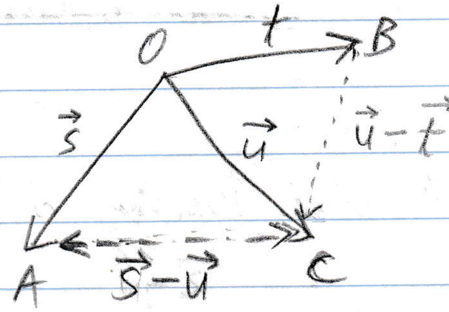
$e + g = 0 \iff e = -g$

$f + h = 0 \iff f = -h$

$\therefore (e,f) = (-g,-h) = -(g,h) = -\vec{v}$

In the figure, $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ represent respectively the vectors $\vec{s}$, $\vec{t}$, and $\vec{u}$.

Verify each statement



(47) $\overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AO} = \vec{0}$

$[\vec{t} + (\vec{u} - \vec{t}) + (\vec{s} - \vec{u}) - \vec{s}]$

$= (\vec{t} - \vec{t}) + (\vec{u} - \vec{u}) + (\vec{s} - \vec{s}) = \vec{0} + \vec{0} + \vec{0} = \vec{0}$

(48) $\overrightarrow{CB} + \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{CA}$

$-(\vec{u} - \vec{t}) - \vec{t} + \vec{s} = \vec{s} - \vec{u}$

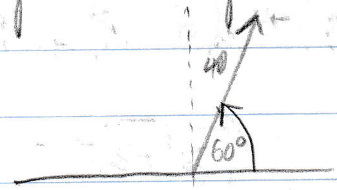
$-\vec{u} + \vec{t} - \vec{t} + \vec{s} = \vec{s} - \vec{u}$

4-4 The Norm of a Vector

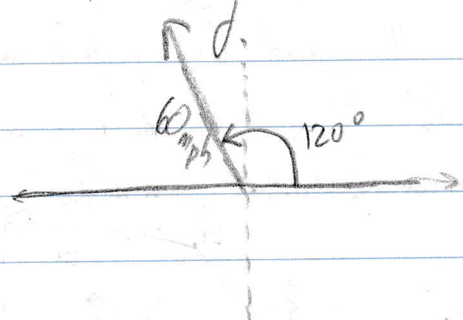
Exercises

Make an accurate drawing to represent the given physical vectors.

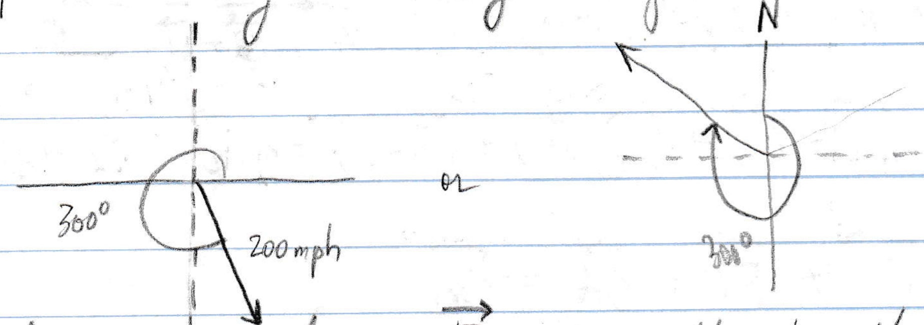
① A force F of 40 lb acting on an object at 60°



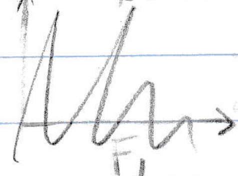
② A car traveling at a velocity $\vec{v}$ of 60 mph at 120°.

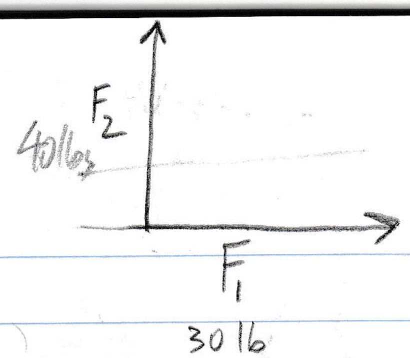


③ A plane traveling at velocity $\vec{v}$ of 200 mph at 300°.



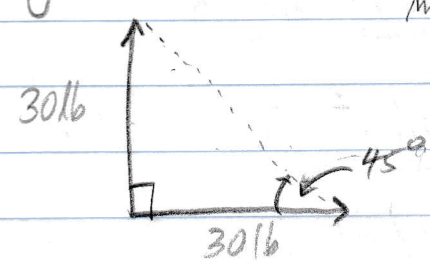
④ A horizontal force $\vec{F}_1$ of 30 lbs to the right and a vertical force $\vec{F}_2$ of 40 lbs upward acting on the same object.





Find the magnitude and direction angle with the horizontal of the resultant in each vector diagram.

5

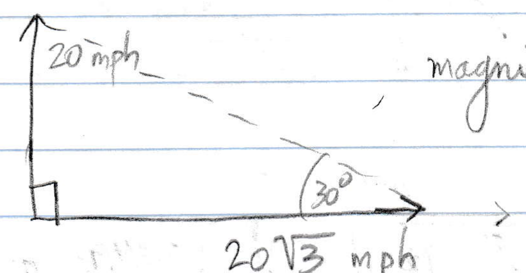


magnitude : $\sqrt{30^2+30^2} = \sqrt{900+900}$

$= \sqrt{1800} = 30\sqrt{2}$

at 45°

6

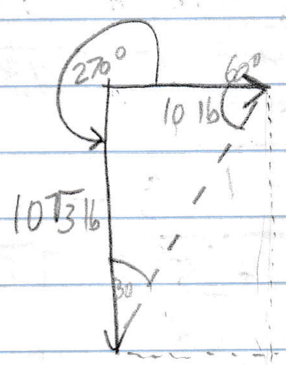


magnitude : $\sqrt{(20^2)+(20\sqrt{3})^2}$

$= \sqrt{400+1200} = \sqrt{1600}$

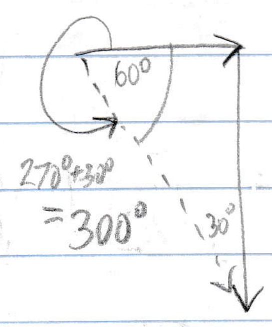
$= 40$ mph at 30°

7

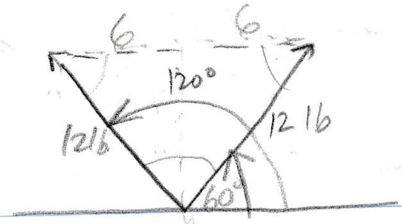


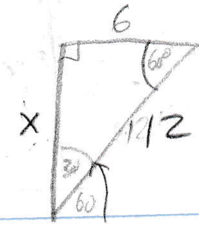
magnitude : $\sqrt{10^2+(10\sqrt{3})^2} = \sqrt{100+300}$

$= \sqrt{400} = 20$ at 300°



(8)

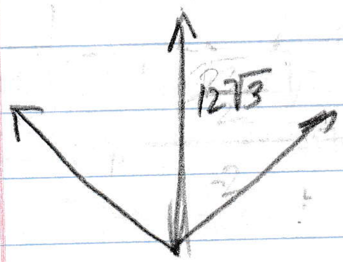




$\sin(60°)$
$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$\sin(60°) = \frac{\sqrt{3}}{2} = \frac{x}{12} \longleftrightarrow x = 6\sqrt{3}$$

$$(-6, 6\sqrt{3}) + (6, 6\sqrt{3}) = (0, 12\sqrt{3})$$



$12\sqrt{3}$ lb at $90°$

{9,...,14} Find the norm of the given vector. (These are too easy.)

(9) $(3,-4)$: $\sqrt{3^2+(-4)^2} = \sqrt{9+16} = 5$

(10) $(-5,12)$: $\sqrt{(-5)^2+12^2} = \sqrt{25+144} = 13$

(11) $(-2,-3)$: $\sqrt{(-2)^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$

(12) $(\sqrt{3}, 1)$: $\sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$

(13) $(\sqrt{3}, \sqrt{5})$: $\sqrt{(\sqrt{3})^2+(\sqrt{5})^2} = \sqrt{3+5} = \sqrt{8}$
$= 2\sqrt{2}$

(14) $(3k, 4k)$: $\sqrt{(3k)^2+(4k)^2} = \sqrt{9k^2+16k^2}$
$= \sqrt{25k^2} = 5k$

{15..18} Find the length of the arrow representation of the vector $\vec{v}$, that is, find $\|\vec{v}\|$.

(15) $\vec{v} = (3\sqrt{2}, 3\sqrt{2})$

$$\|\vec{v}\| = \sqrt{(3\sqrt{2})^2 + (3\sqrt{2})^2} = \sqrt{18+18} = \sqrt{36} = 6$$

(16) $\vec{v} = (\sqrt{3}+1, \sqrt{3}-1)$

$$\|\vec{v}\| = \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2} = \sqrt{(3+2\sqrt{3}+1)+(3-2\sqrt{3}+1)}$$

$$= \sqrt{(4+2\sqrt{3})+(4-2\sqrt{3})} = \sqrt{8} = 2\sqrt{2}$$

(17) $\vec{v} = (1,1) + (2,3) = (3,4)$

$$\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{9+16} = 5$$

(18) $\vec{v} = (5,2) - (2,1) = (3,1)$

$$\|\vec{v}\| = \sqrt{3^2+1^2} = \sqrt{9+1} = \sqrt{10}$$

Given $\vec{r} = (2,3)$, $\vec{s} = (-3,1)$, and $\vec{t} = (4,-2)$.
Find each of the following scalars.

(19) $\|\vec{r}+\vec{s}\| = \|(2,3)+(-3,1)\| = \|(-1,4)\|$

$$= \sqrt{(-1)^2+4^2} = \sqrt{1+16} = \sqrt{17}$$

(20) $\|\vec{r}-\vec{s}\| = \|(2,3)-(-3,1)\| = \|(5,2)\|$

$= \sqrt{5^2+2^2} = \sqrt{25+4} = \sqrt{29}$

(21) $\|\vec{t}-\vec{s}\| = \|(4,-2)-(-3,1)\| = \|(7,-3)\|$

$= \sqrt{7^2+(-3)^2} = \sqrt{49+9} = \sqrt{58}$

(22) $\|\vec{r}+\vec{s}+\vec{t}\| = \|(2,3)+(-3,1)+(4,-2)\|$

$= \|(2-3+4, 3+1-2)\| = \|(3,2)\|$

$= \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

(23) $\|\vec{r}+\vec{s}-\vec{t}\| = \|(2,3)+(-3,1)-(4,-2)\|$

$= \|(2-3-4, 3+1+2)\| = \|(-5,6)\|$

$= \sqrt{(-5)^2+6^2} = \sqrt{25+36} = \sqrt{61}$

(24) $\|\vec{r}-\vec{s}-\vec{t}\| = \|(2,3)-(-3,1)-(4,-2)\|$

$= \|(2+3-4, 3-1+2)\| = \|(1,4)\|$

$= \sqrt{1^2+4^2} = \sqrt{17}$

Verify the triangle inequality for the vectors $\vec{r}$ and $\vec{s}$.

25 $\vec{r} = (5,2)$; $\vec{s} = (2,1)$

$$\|\vec{r}+\vec{s}\|^2 \leq (\|\vec{r}\| + \|\vec{s}\|)^2$$

$$\|(5,2)+(2,1)\|^2 \leq \left(\sqrt{5^2+2^2} + \sqrt{2^2+1^2}\right)^2$$

$$\|(7,3)\|^2 \leq \left(\sqrt{25+4} + \sqrt{4+1}\right)^2$$

$$\left(\sqrt{7^2+3^2}\right)^2 \leq \left(\sqrt{29}+\sqrt{5}\right)^2$$

$$\left(\sqrt{58}\right)^2 \leq 29 + \sqrt{145} + \sqrt{145} + 5$$

$$58 \leq 34 + 2\sqrt{145}$$

$$34 + 2\sqrt{144} = 34 + 2(12) = 58 < 34 + 2\sqrt{145}$$

26 $\vec{r} = (3,1)$; $\vec{s} = (2,-1)$

$$\|\vec{r}+\vec{s}\|^2 \leq (\|\vec{r}\| + \|\vec{s}\|)^2$$

$$\|(3,1)+(2,-1)\|^2 \leq \left(\sqrt{3^2+1^2} + \sqrt{2^2+(-1)^2}\right)^2$$

$$\|(5,0)\|^2 \leq \left(\sqrt{10} + \sqrt{5}\right)^2$$

$$\left(\sqrt{25}\right)^2 = 25 \leq 10 + 2\sqrt{50} + 5 = 15 + 2\cdot 5\sqrt{2}$$

$25 \leq 15 + 10\sqrt{2}$ | $15 + 2\sqrt{49} = 15 + 2\cdot 7 = 29$
$< 15 + 2\sqrt{50}$ and $25 < 29$

(27) $\vec{r} = (0,-2)$; $\vec{s} = (3,4)$

$$\|\vec{r}+\vec{s}\|^2 = \|(0,-2)+(3,4)\|^2 = \|(3,2)\|^2$$

$$= \left(\sqrt{3^2+2^2}\right)^2 = \left(\sqrt{13}\right)^2 = 13$$

$$\left(\|\vec{r}\|+\|\vec{s}\|\right)^2 = \left(\sqrt{(-2)^2} + \sqrt{3^2+4^2}\right)^2$$

$$= \left(\sqrt{4}+\sqrt{25}\right)^2 = 7^2 = 49$$

$$13 < 49$$

(28) $\vec{r} = (-1,-1)$; $\vec{s} = (3,2)$

$$\|\vec{r}+\vec{s}\|^2 = \|(2,1)\|^2 = \left(\sqrt{2^2+1^2}\right)^2 = 5$$

$$\left(\|\vec{r}\|+\|\vec{s}\|\right)^2 = \left(\sqrt{(-1)^2+(-1)^2} + \sqrt{3^2+2^2}\right)^2$$

$$= \left(\sqrt{2}+\sqrt{13}\right)^2 = 2 + 2\sqrt{26} + 13 = 15 + 2\sqrt{26}$$

$$15 + 2\sqrt{26} > 15 + 2\sqrt{25} = 15 + 2 \cdot 5 = 25 > 5$$

(29) Prove: $\|-\vec{v}\| = \|\vec{v}\|$ for every vector $\vec{v}$.

$$\|-(v_1, v_2)\| = \|(-v_1, -v_2)\| = \sqrt{(-v_1)^2 + (-v_2)^2}$$

$$= \sqrt{v_1^2 + v_2^2} = \|\vec{v}\|$$

(30) Prove: $\|\vec{v}\| \geq 0$ for every vector $\vec{v}$.

$$\|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2} \geq 0$$

(31) Prove: $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$

$$\|(0,0)\| = \sqrt{0^2+0^2} = \sqrt{0} = 0$$

if $\|\vec{v}\| = 0$, then $\sqrt{v_1^2 + v_2^2} = 0$

so $v_1^2 + v_2^2 = 0 \Longleftrightarrow v_1^2 = -v_2^2 \quad \therefore v_1 = 0 = v_2$

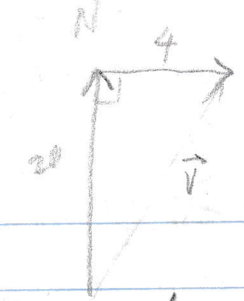
and $\vec{v} = (0,0) = \vec{0}$

(32) Prove: if $\vec{v_1} = (a, b)$ and $\vec{v_2} = (b, a)$, then $\|\vec{v_1}\| = \|\vec{v_2}\|$

$$\|\vec{v_1}\| = \sqrt{a^2+b^2} = \sqrt{b^2+a^2} = \|\vec{v_2}\|$$

{33,, 36} Make an accurate scale vector drawing. Where possible, make use of special right-triangle relationships. In other cases give an approximate answer on the basis of your drawing.

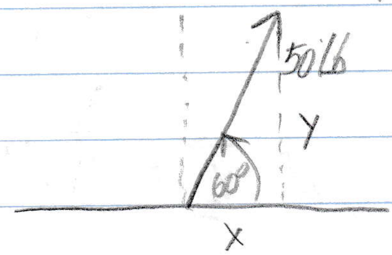
(33) A boat sets out to travel north at 20 mph. A wind from the west moves the boat eastward at 4 mph. Determine the boat's velocity along the path it travels.



$$\|\vec{v}\| = \sqrt{20^2+4^2} = \sqrt{416}$$
$$= 2\sqrt{104} = 2\cdot 2\sqrt{26} = 4\sqrt{26} \text{ mph}$$

$$\arctan\left(\tfrac{20}{4}\right) = \arctan(5) \approx 78.6^\circ$$

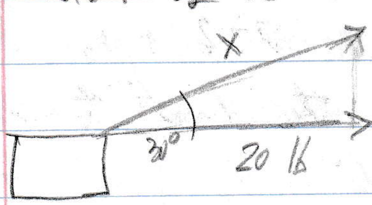
(34) Suppose a force of 50 lb is exerted on an object in a direction of 60° from the horizontal. What two forces, one in a horizontal direction and one in a vertical direction, would have the same effect on the object?



$$\sin\left(\tfrac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \frac{Y}{50}$$
$$Y = 25\sqrt{3}$$
$$\cos\left(\tfrac{\pi}{3}\right) = \frac{1}{2} = \frac{X}{50}$$
$$X = 25$$

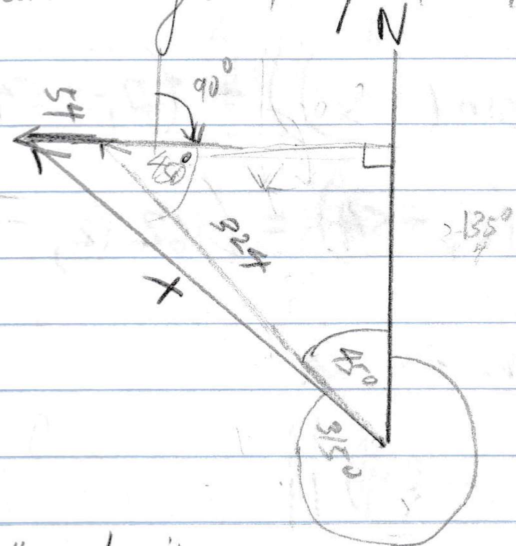
$\therefore$ 25 lb horizontally
$25\sqrt{3}$ lb vertically

(35) A trunk is dragged along a horizontal plane by a rope which makes an angle of 30° with the horizontal. What force must be exerted on the rope to give the same result as a horizontal force of 20 lb?



$$\cos\left(\tfrac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = \frac{20}{X}$$
$$X = \frac{40}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{40\sqrt{3}}{3} \text{ lb.}$$

(36) A plane flies a course of 315° at 324 mph. If a 54 mph wind is blowing from 90°, determine the plane's ground speed and course.



$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{x}{324} \longleftrightarrow x = 162\sqrt{2}$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{y}{324} \longleftrightarrow y = 162\sqrt{2}$

Going 315° clockwise with North direction as horizontal, WEIRD

let $324 = 1$ unit

then $54 = \frac{54}{324}$ unit $= \frac{1}{6}$

resultant vector $\frac{9}{8}''$ $\quad \frac{\frac{9}{8}}{x} = \frac{1}{324}$, $x = 364.5 \approx 365$

direction 310° $\quad$ This solution is not clear.

assuming east is positive horizontal x, north is positive vertical y, break down the plane's velocity into a vector in the x-y plane.

The plane is traveling 324 mph at a bearing 315° degrees: $(324\cos(315°), 324 \cdot \sin(315°))$

The wind is blowing from the north (to the south) at 54 mph or $(0, -54)$.

The actual ground speed would be the vector sum of these two vectors.

$$(324 \cdot \cos(315°), 324 \sin(315°)) + (0, -54)$$

$$(162\sqrt{2}, -162\sqrt{2}) + (0, -54) = (162\sqrt{2}, -162\sqrt{2} - 54)$$

The magnitude of this vector is

$$\| (162\sqrt{2}, -162\sqrt{2} - 54) \|$$

$$= \sqrt{(162\sqrt{2})^2 + (-162\sqrt{2} - 54)^2}$$

$$= \sqrt{52488 + 17496\sqrt{2} + 55404}$$

$$= 54\sqrt{6\sqrt{2} + 37} \approx 364.191$$

For direction: $\tan(\alpha) = \frac{-162\sqrt{2} + 54}{162\sqrt{2}}$

$$\tan(\alpha) = \frac{\sqrt{2}+6}{6} \longleftrightarrow \alpha \approx 0,8904 \approx 51° ?$$

$$\tan(\alpha) = \frac{-162\sqrt{2} - 54}{162\sqrt{2}}$$

$$\alpha = -\arctan\left(\frac{\sqrt{2}+6}{6}\right) = -0,8904 \approx -51°$$

$$360° - 51° = 309°$$

37

By

38

(37) Prove: For any two vectors $\vec{r}$ and $\vec{t}$,

$$\|\vec{r}\| - \|\vec{t}\| \leq \|\vec{r} - \vec{t}\|$$

(Hint: Apply the triangle inequality with $\vec{v} = \vec{r} - \vec{t}$)

By triangle inequality, $\|\vec{v} + \vec{t}\| \leq \|\vec{v}\| + \|\vec{t}\|$

let $\vec{v} = \vec{r} - \vec{t}$

$$\|\vec{r} - \vec{t} + \vec{t}\| \leq \|\vec{r} - \vec{t}\| + \|\vec{t}\|$$

$$\|\vec{r}\| - \|\vec{t}\| \leq \|\vec{r} - \vec{t}\|$$

(38) Prove: For any three vectors, $\vec{r}$, $\vec{s}$, and $\vec{t}$

$$\|\vec{r} - \vec{s}\| + \|\vec{s} - \vec{t}\| \geq \|\vec{r} - \vec{t}\|$$

$$\|\vec{r} - \vec{t}\| = \|(\vec{r} - \vec{s}) + (\vec{s} - \vec{t})\| \leq \|\vec{r} - \vec{s}\| + \|\vec{s} - \vec{t}\|$$

4-5 Parallel & Perpendicular Vectors

Multiplication of a Vector by a scalar.

Given $\vec{v} = (3,-2)$, find the following scalar multiples of $\vec{v}$.

1. $2\vec{v} = (6,-4)$ TOO easy

Given $\vec{s} = (-3,4)$. Find the vector norms.

(5) $\|2\vec{s}\| = \sqrt{(-6)^2+(8)^2} = \sqrt{36+64} = 10$

(6) $\|4\vec{s}\| = \sqrt{(-12)^2+(16)^2} = 20$

(7) $\|-3\vec{s}\| = \sqrt{(9)^2+(-12)^2} = 15$

(8) $\|\frac{1}{3}\vec{s}\| = \sqrt{(-1)^2+\left(\frac{4}{3}\right)^2} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$

{9..12} Where possible, find k such that $\vec{v} = k\vec{t}$

(9) $\vec{v} = (6,3)$ and $\vec{t} = (2,1)$

$(6,3) = k(2,1) \longleftrightarrow 2k=6$ and $k=3$

$\therefore k=3$

(10) $\vec{v} = (2,\sqrt{3})$ and $\vec{t} = (\sqrt{2},\sqrt{6})$

$(2,\sqrt{3}) = k(\sqrt{2},\sqrt{6}) \longleftrightarrow \sqrt{2}k=2$ and $\sqrt{6}k=\sqrt{3}$

$k=\sqrt{2}$ and $k=\frac{1}{\sqrt{2}}$ $\therefore$ no solution.

$\sqrt{2} \neq \frac{\sqrt{2}}{2}$

11

12

11 $\vec{v} = (4, -6)$ and $\vec{t} = (-6, 10)$

$(4,-6) = k(-6,10) \longleftrightarrow -6k = 4$ and $10k = -6$

$k = -\frac{2}{3}$ and $k = -\frac{3}{5}$ but $-\frac{2}{3} \neq -\frac{3}{5}$

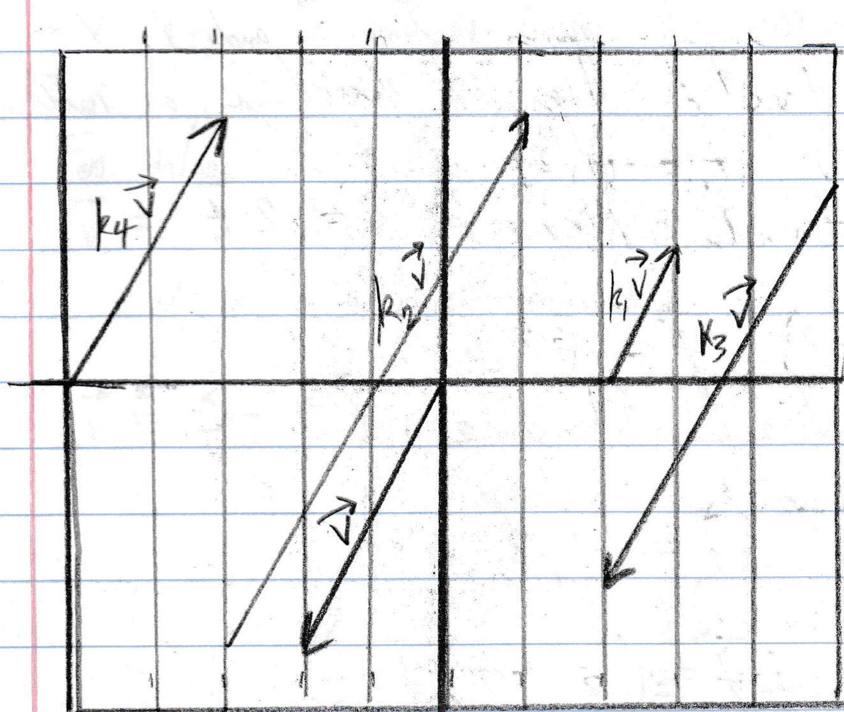
$\therefore$ no solution

12 $\vec{v} = (-\sqrt{3}, 4\sqrt{3})$ and $\vec{t} = (-1, 4)$

$(-\sqrt{3}, 4\sqrt{3}) = k(-1, 4)$

$-k = -\sqrt{3} \longleftrightarrow k = \sqrt{3}$

$4k = 4\sqrt{3} \longleftrightarrow k = \sqrt{3} \quad \therefore \quad k = \sqrt{3}$



13 Find k_1, k_2, k_3, k_4

14 Find the norm of each vector represented.

First, we find $\vec{v} = (-2, -4)$. Since $k_1\vec{v} = (1, 2)$

$k_1 = \frac{(1, 2)}{(-2, -4)} = -\frac{1}{2}$; $k_2\vec{v} = (4, 8)$ so $k_2 = -2$

$k_3\vec{v} = (-3, -6)$ so $k_3 = \frac{-3}{-2} = \frac{-6}{-4} = \frac{3}{2}$

$k_4\vec{v} = (2, 4)$ so $k_4 = -1$

(14) $k_1\vec{v} = \left|-\frac{1}{2}\right| \, \|(-2,-4)\|$

$= \frac{1}{2}\cdot\sqrt{2^2+4^2} = \frac{1}{2}\sqrt{4+16} = \frac{\sqrt{20}}{2} = \sqrt{5}$

(15) $k_2\vec{v} = |-2| \, \|(-2,-4)\| = -2\sqrt{20} = \cdot 4\sqrt{5}$

$k_3\vec{v} = \left|\frac{3}{2}\right| \, \|(-2,-4)\| = \frac{3}{2}\sqrt{20} = 3\sqrt{5}$

$k_4\vec{v} = |-1| \cdot \|(-2,-4)\| = \cdot\sqrt{20} = \cdot 2\sqrt{5}$

{15, 20} Tell whether the given vectors $\vec{s}$ and $\vec{t}$ have the same direction, opposite directions, or neither.

(15) $\vec{s} = (2,4)$, $\vec{t} = (1,2)$

Same direction because $\vec{s} = 2\cdot\vec{t}$

(16) $\vec{s} = (1,-3)$, $\vec{t} = (-2,6)$

Opposite direction because $\vec{t} = -2\vec{s}$

(17) $\vec{s} = (3,1)$; $\vec{t} = (1,3)$

neither.

(18) $\vec{s} = (-1,6)$; $\vec{t} = \left(\frac{2}{3},-4\right)$

opposite direction because $\vec{t} = -\frac{2}{3}\vec{s}$

(19) $\vec{s} = (2,1)$; $\vec{t} = (-1,-2)$

neither

(20) $\vec{s} = (2,\sqrt{6})$; $\vec{t} = (\sqrt{2}, 3\sqrt{2})$

neither

Describe in words the coordinate-plane representation of the set of vectors specified by the following. Assume $k \in \mathbb{R}$, and all arrows involved are in standard position.

(21) $\{(x,y) : (x,y) = k(2,3)\}$

The set of vectors $(2k, 3k) \in \mathbb{R} \times \mathbb{R}$

A line through the origin and $(2,3)$

(22) $\{(x,y) : (x,y) = k\sqrt{3}\,(\sqrt{2}, 2\sqrt{3})\}$

A line through the origin and $(\sqrt{2}, 2\sqrt{3})$.

For what $k \in \mathbb{R}$ will $\vec{v}$ and $\vec{u}$ be parallel vectors?

(23) $\vec{v} = (2,5)$, $\vec{u} = (6,k)$

if $k = 15$, then $\vec{u} = 3\vec{v}$

(24) $\vec{v} = (4,-6)$; $\vec{u} = (k,3)$

if $k = -2$ then $\vec{u} = -\frac{1}{2}\vec{v}$

(25) $\vec{v} = (a,b)$, $\vec{u} = \left(k, -\frac{b}{a}\right)$

if $k = -1$ then $\vec{u} = -\frac{1}{a}\vec{v}$

(26) $\vec{v} = (2, \sqrt{3})$; $\vec{u} = (k, -6)$

THOUGHT PROCESS: $\frac{-6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-6\sqrt{3}}{3} = -2\sqrt{3}$

and $-2\sqrt{3}(2) = -4\sqrt{3}$ so $k = -4\sqrt{3}$

because $\vec{u} = -2\sqrt{3}\,\vec{v}$

Find (a) the unit vector in the same direction as $\vec{t}$
(b) the unit vector in the opposite direction as $\vec{t}$

(27) (a) $\vec{t} = (3,4)$

The unit vector is $\dfrac{\vec{t}}{\|\vec{t}\|} = \dfrac{(3,4)}{\sqrt{3^2+4^2}} = \dfrac{(3,4)}{5}$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$ in same direction

(b) $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ in opposite direction.

(28) $\vec{t} = (-2,3)$

(a) $\dfrac{\vec{t}}{\|\vec{t}\|} = \dfrac{(-2,3)}{\sqrt{(-2)^2+3^2}} = \dfrac{(-2,3)}{\sqrt{13}} = \left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$

$= \left(\frac{-2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right)$ in same direction

(b) $\left(\frac{2\sqrt{13}}{13}, -\frac{3\sqrt{13}}{13}\right)$ in opposite direction.

(29) $\vec{t} = (-\sqrt{3}, 1)$

(a) $\dfrac{(-\sqrt{3}, 1)}{\sqrt{(-\sqrt{3})^2+1^2}} = \dfrac{(-\sqrt{3}, 1)}{2} = \left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$

(b) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

(30) $\vec{t} = (\sqrt{2}, -\sqrt{7})$

(a) $$\frac{(\sqrt{2}, -\sqrt{7})}{\sqrt{(\sqrt{2})^2 + (-\sqrt{7})^2}} = \frac{(\sqrt{2}, -\sqrt{7})}{\sqrt{2+7}} = \left(\frac{\sqrt{2}}{3}, \frac{-\sqrt{7}}{3}\right)$$

(b) $$\left(\frac{-\sqrt{2}}{3}, \frac{\sqrt{7}}{3}\right)$$

Given that $\vec{v}$ and $\vec{t}$ are elements of $R \times R$ and r and s are real numbers, Prove the following properties of the multiplication of a vector by a scalar!

(31) Closure : $r \cdot \vec{v} \in R \times R$

$r\vec{v} = (r \cdot v_1, r \cdot v_2)$

Since $r \cdot v_1$ and $r \cdot v_2$ are real numbers, $r\vec{v} \in R \times R$

(32) Property of -1 : $(-1)\vec{v} = -\vec{v}$

$(-1)\vec{v} = -(v_1, v_2) = (-v_1, -v_2) = -\vec{v}$

(33) Associative : $(r \cdot s)\vec{v} = r(s\vec{v})$

$(r \cdot s)\vec{v} = ((r \cdot s)v_1, (r \cdot s) \cdot v_2) = (r(s \cdot v_1), r(s \cdot v_2))$

$= r(s \cdot \vec{v})$

(34) Identity Element : $1\vec{v} = \vec{v}$

$1\vec{v} = (v_1, v_2) = \vec{v}$

If $\vec{s} = (5,2)$ and $\vec{t} = (-4,3)$, determine the indicated vector.

(35) $2(\vec{s}+\vec{t}) = 2(5-4, 2+3) = (2, 10)$

(36) $3\vec{s} - 2\vec{t} = 3(5,2) - 2(-4,3)$

$= (15, 6) - (-8, 6)$

$= (15, 6) + (8, -6) = (23, 0)$

(37) $-2\vec{s} + 3\vec{t} = -2(5,2) + 3(-4, 3)$

$= (-10, -4) + (-12, 9)$

$= (-22, 5)$

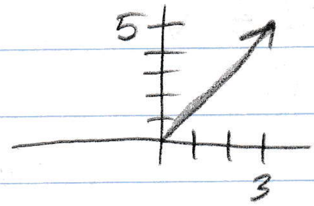
(38) $\frac{1}{3}\vec{s} + \frac{2}{3}\vec{t} = \frac{1}{3}(5,2) + \frac{2}{3}(-4,3)$

$= \left(\frac{5}{3}, \frac{2}{3}\right) + \left(-\frac{8}{3}, 2\right)$

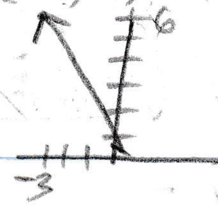
$= \left(-1, \frac{8}{3}\right)$

Simplify each vector sum and represent the result as an arrow in standard position in the coordinate plane.

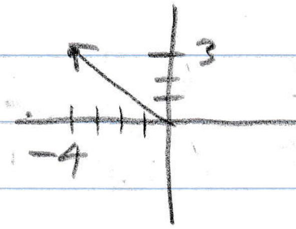
(39) $5(0,1) + 3(1,0) = (0,5) + (3,0) = (3,5)$



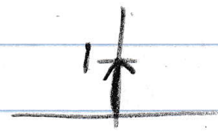
(40) $2(-3,4) + 1(3,-2) = (-6,8) + (3,-2)$
$= (-3,6)$



(41) $2(1,3) + (-3)(2,1) = (2,6) + (-6,-3)$
$= (-4,3)$



(42) $0(1,0) + 1(0,1) = (0,0) + (0,1) = (0,1)$



For what value of x and y is the given statement true?

(43) $x(2,3) + y(3,-1) = (14,10)$

$(2x, 3x) + (3y, -y) = (14,10)$

$$\begin{aligned} 2x + 3y &= 14 \\ 3x - y &= 10 \end{aligned} \longleftrightarrow \begin{aligned} 2x + 3y &= 14 \\ 9x - 3y &= 30 \\ \hline 11x \quad &= 44 \end{aligned}$$

$x = 4$ so $2(4) + 3y = 14 \longleftrightarrow 3y = 6$
$y = 2$

(44) $x(3,1) + y(2,3) = (0,7)$

$$\begin{aligned} 3x + 2y &= 0 \\ x + 3y &= 7 \end{aligned} \longleftrightarrow \begin{aligned} 3x + 2y &= 0 \\ -3x - 9y &= -21 \\ \hline -7y &= -21 \end{aligned}$$

$y = 3$
$3x + 6 = 0$
$3x = -6$
$x = -2$

(45) $x(1,2) + y(2,4) = (6,12)$

$(x, 2x) + (2y, 4y) = (6,12)$

$$\begin{aligned} x + 2y &= 6 \\ 2x + 4y &= 12 \end{aligned} \longleftrightarrow \begin{aligned} -2x - 4y &= -12 \\ 2x + 4y &= 12 \\ \hline 0 + 0 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

For any $x \in \mathbb{R}$ and $y = \frac{1}{2}(6-x)$

How? There are an infinite amount of solutions because the equations are linear dependent. That is, the second equation is exactly two times the first equation so can be eliminated.

Hence, solving for y: $x + 2y = 6$

$2y = 6 - x$

$y = \frac{1}{2}(6-x)$

(46) $x(3,6) + y(1,2) = (7,10)$

$(3x, 6x) + (y, 2y) = (7,10)$

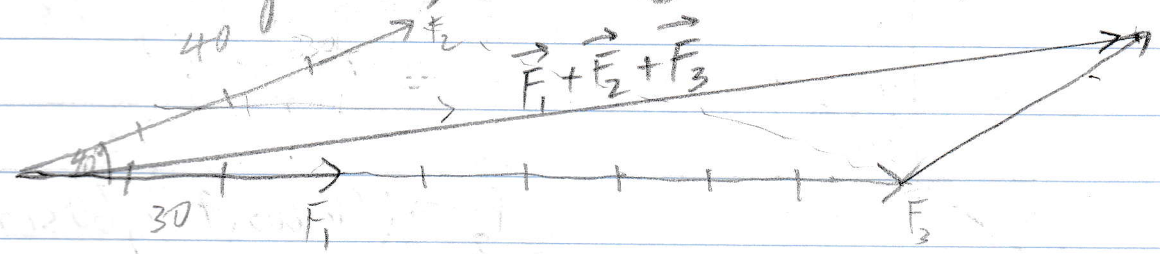
$$\begin{aligned} 3x + y &= 7 \\ 6x + 2y &= 10 \end{aligned} \longleftrightarrow \begin{aligned} -6x - 2y &= -14 \\ 6x + 2y &= 10 \\ \hline 0 &= -4 \end{aligned}$$

No solutions.

$$\left.\begin{aligned} y &= 7 - 3x \\ 2y &= 10 - 6x \to y = 5 - 3x \end{aligned}\right\} \text{which is impossible so no values satisfy.}$$

On a scale drawing, estimate the magnitude and direction of the resultant of three forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$.

(47) $\vec{F}_1$ is a horizontal force of 30 lb to the right, $\vec{F}_2$ is a force of 40 lb acting in a counterclockwise direction 40° from the direction of $\vec{F}_1$, and $\vec{F}_3 = 2\vec{F}_1$.



$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (30\cdot\cos(0), 30\cdot\sin(0)) + (40\cdot\cos(40°), 40\sin(40°)) + (60\cos(0), 60\sin(0))$$

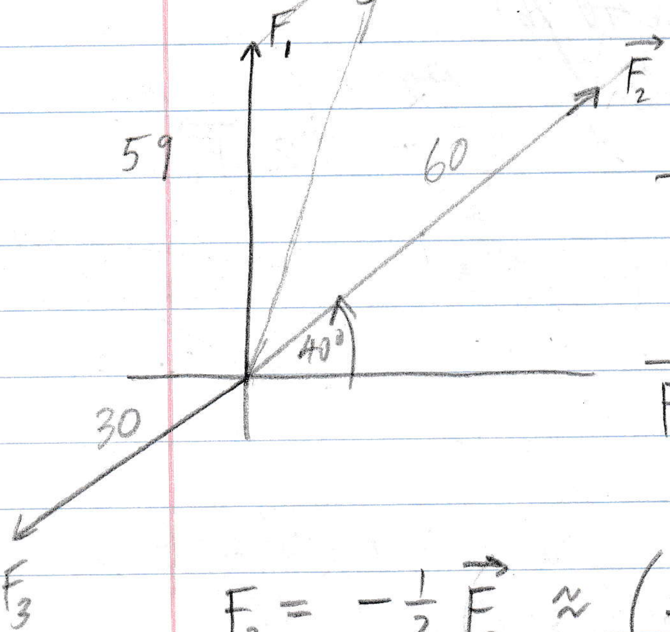
$$= (30, 0) + (30.64178, 25.7115) + (60, 0)$$
$$= (120.6418, 25.7115)$$

$$\alpha = \arctan\left(\frac{25.7115}{120.6418}\right) \approx 0.2099811 \text{ radians} \approx 12°$$

Magnitude of $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \|(120.6418, 25.7115)\|$

$$= \sqrt{(120.6418)^2 + (25.7115)^2} \approx 123.3512$$

(48) $\vec{F_1}$ is a vertical force upward of 59 tons, $\vec{F_2}$ is a force of 60 tons acting in a clockwise direction $40°$ from the direction of $\vec{F_1}$, and $\vec{F_3} = -\frac{1}{2}\vec{F_2}$.



$$\vec{F_1} = \left(59\cos\left(\tfrac{\pi}{2}\right), 59\sin\left(\tfrac{\pi}{2}\right)\right) = (0, 59)$$

$$\vec{F_2} = (60\cos(40°), 60\sin(40°)) \approx (45.96267, 38.56726)$$

$$F_3 = -\frac{1}{2}\vec{F_2} \approx (-22.98134, -19.28363)$$

$$\therefore \vec{F_1} + \vec{F_2} + \vec{F_3} = (22.98134, 78.28363)$$

$$\|\vec{F_1} + \vec{F_2} + \vec{F_3}\| = \sqrt{(22.98134)^2 + (78.28363)^2}$$

$$\approx 81.58718 \approx 82 \text{ tons}$$

$$\alpha = \arctan\left(\frac{78.28363}{22.98134}\right) \approx 1.285254 \text{ radians}$$

$$\approx 1.285254 \text{ rad}\left(\frac{180°}{\pi \text{ rad}}\right) \approx 73.6396°$$

$$\approx 74°$$

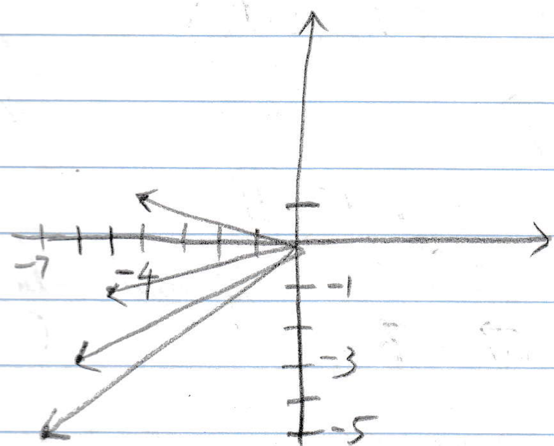
Let $\vec{v}$ and $\vec{u}$ be the given vectors and
$L = \{ \vec{v} + k\vec{u} : k \in \{0, 1, 2, 3\} \}$.

Draw standard representations of the elements of L.

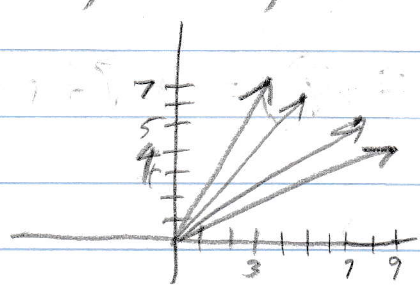
(49) $\vec{v} = (-4, 1)$, $\vec{u} = (-1, -2)$

$L = \{ (-4,1) + 0, (-4,1) + (-1,-2), (-4,1) + 2(-1,-2),$
$(-4,1) + 3(-1,-2) \}$

$= \{ (-4,1), (-5,-1), (-6,-3), (-7,-5) \}$



(50) $\vec{v} = (3, 7)$; $\vec{u} = (2, -1)$
$L = \{ (3,7), (3,7) + (2,-1), (3,7) + 2(2,-1),$
$(3,7) + 3(2,-1) \}$
$= \{ (3,7), (5,6), (7,5), (9,4) \}$



Given $\vec{v}$ and $\vec{t}$ are elements of $R \times R$ and r and s are elements of R. Prove the following properties of the multiplication of a vector by a scalar.

(51) Substitution: if $r = s$ and $\vec{v} = \vec{t}$, then $r \cdot \vec{v} = s \cdot \vec{t}$

Since $\vec{v} = \vec{t}$, $v_1 = t_1$ and $v_2 = t_2$
Since $r = s$, $r \cdot v_1 = st_1$ and $r \cdot v_2 = st_2$
$\therefore$ $(r \cdot v_1, r \cdot v_2) = (s \cdot t_1, s \cdot t_2)$

so $r \cdot \vec{v} = s \cdot \vec{t}$

(52) Zero product: $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$ or $\vec{v} = \vec{0}$.

If $r = 0$, $r \cdot \vec{v} = 0 \cdot \vec{v} = (0 \cdot v_1, 0 \cdot v_2) = (0,0) = \vec{0}$
If $\vec{v} = \vec{0} = (0,0)$, $r \cdot \vec{v} = (r \cdot 0, r \cdot 0) = (0,0) = \vec{0}$

If $r \cdot \vec{v} = \vec{0}$, $(r \cdot v_1, r \cdot v_2) = (0,0)$, $r \cdot v_1 = 0$
$r \cdot v_2 = 0$, $r = 0$ or $v_1 = 0$ and $v_2 = 0$

(53) Distributive: $r(\vec{v} + \vec{t}) = r\vec{v} + r \cdot \vec{t} = \vec{v} \cdot r + \vec{t} \cdot r$

$$r(\vec{v}+\vec{t}) = r(v_1+t_1, v_2+t_2) = (r(v_1+t_1), r(v_2+t_2))$$

$$= (r\cdot v_1 + r\cdot t_1, r\cdot v_2 + r\cdot t_2) = (r\cdot v_1, r\cdot v_2) + (r t_1, r t_2)$$

$$= r\cdot\vec{v} + r\cdot\vec{t} = \vec{v}\cdot r + \vec{t}\cdot r$$

54) *Distributive*: $(r+s)\vec{v} = r\vec{v} + s\vec{v} = \vec{v}\cdot r + \vec{v}s$

$$(r+s)\vec{v} = ((r+s)v_1, (r+s)v_2) = (rv_1 + sv_1, rv_2 + sv_2)$$

$$= (r\cdot v_1, r\cdot v_2) + (s\cdot v_1, s\cdot v_2) = r\cdot\vec{v} + s\cdot\vec{v}$$

$$= \vec{v}\cdot r + \vec{v}\cdot s$$

55) $r(\vec{v}-\vec{t}) = r\cdot\vec{v} - r\vec{t} = \vec{v}\cdot r - \vec{t}\cdot r$

$$r(\vec{v}-\vec{t}) = r(\vec{v}+(-\vec{t})) = r\vec{v} + r(-\vec{t})$$

$$r\cdot\vec{v} + (-r\vec{t}) = r\vec{v} - r\vec{t} = r\cdot\vec{v} + (r\cdot(-\vec{t}))$$

$$= \vec{v}\cdot r + (-\vec{t})r = \vec{v}\cdot r - \vec{t}\cdot r$$

56) $(r-s)\vec{v} = r\cdot\vec{v} - s\vec{v} = \vec{v}\cdot r - \vec{v}\cdot s$

$$(r-s)\vec{v} = r\cdot\vec{v} - s\vec{v} = r\cdot\vec{v} + (-s\vec{v})$$

$$= r\vec{v} + (-s)\vec{v} = r\vec{v} + s(-\vec{v}) = \vec{v}r - \vec{v}s$$

(57) Prove: $\|\vec{v}+\vec{t}\| = \|\vec{t}+\vec{v}\|$

$\|\vec{v}+\vec{t}\| = \|(v_1+t_1, v_2+t_2)\|$

$= \sqrt{(v_1+t_1)^2+(v_2+t_2)^2}$

$\|\vec{t}+\vec{v}\| = \|(t_1+v_1, t_2+v_2)\|$

$= \sqrt{(t_1+v_1)^2+(t_2+v_2)^2}$

Since addition is commutative in $\mathbb{R}$,

$\|\vec{v}+\vec{t}\| = \|\vec{t}+\vec{v}\|$

KEY STEP

(58) Let $\vec{t}$ and $\vec{v}$ be parallel vectors.
Prove: If $\vec{v}=\vec{0}$, then $\vec{v}$ is a scalar multiple of $\vec{t}$; if $\vec{v}\neq\vec{0}$, then $\vec{t}$ is a scalar multiple of $\vec{v}$

If $\vec{v}=\vec{0}$ then $\vec{v}=(0,0)=(0\cdot t_1, 0\cdot t_2)$
$= 0\cdot\vec{t}$ and $\vec{v}$ is a scalar multiple of $\vec{t}$.

If $\vec{v}\neq\vec{0}$ and since $\vec{v}$ and $\vec{t}$ are parallel, either $\vec{v}=k\vec{t}$, $\vec{t}=\frac{1}{k}\vec{v}$ (where $k=0$, $\vec{t}\neq\vec{0}$),
or $\vec{t}=\vec{0}=(0,0)=(0\cdot v_1, 0\cdot v_2)=0\cdot\vec{v}$

(59) Prove: $\vec{v}$ and $\vec{t}$ have the same direction if and only if $\|\vec{v}+\vec{t}\| = \|\vec{v}\| + \|\vec{t}\|$

SQUARE BOTH SIDES

If $\|\vec{v}+\vec{t}\| = \|\vec{v}\| + \|\vec{t}\|$,

$$(v_1+t_1)^2 + (v_2+t_2)^2 = v_1^2 + v_2^2 + t_1^2 + t_2^2 + 2v_1t_1$$
$$+ 2v_2t_2 = v_1^2 + v_2^2 + t_1^2 + t_2^2 + 2\sqrt{(v_1^2+v_2^2)(t_1^2+t_2^2)}$$

KEY STEP $$\boxed{2v_1t_1 + 2v_2t_2 = 2\sqrt{v_1^2t_1^2 + v_2^2t_2^2 + v_1^2t_2^2 + v_2^2t_1^2}},$$

$$v_1^2t_1^2 + v_2^2t_2^2 + 2v_1v_2t_1t_2 = v_1^2t_1^2 + v_2^2t_1^2 + v_1^2t_2^2 + v_2^2t_2^2,$$

$$2v_1v_2t_1t_2 = v_2^2t_1^2 + v_1^2t_2^2, \quad 0 = (v_2t_1 - v_1t_2)^2$$

$$v_2t_1 = v_1t_2, \quad v_1 = \left(\frac{v_2}{t_2}\right)t_1, \quad \vec{v} = \left(\left(\frac{v_2}{t_2}\right)t_1, \left(\frac{v_2}{t_2}\right)t_2\right)$$

$$\vec{v} = \left(\frac{v_2}{t_2}\right)\vec{t} \qquad \frac{v_2}{t_2} > 0 \text{ if } v_2, t_2 > 0$$

$$\therefore \frac{v_2}{t_2} \text{ and } v_2t_2 > 0$$

Since $\vec{v}$ is a positive scalar multiple of $\vec{t}$, they have the same direction.

Conversely, if $\vec{v}$ and $\vec{t}$ have same direction $\vec{v} = k\vec{t}$ where $k > 0$.

$$\|\vec{v}+\vec{t}\| = \|k\vec{t}+\vec{t}\| = \|(k+1)\vec{t}\|$$
$$= (k+1)\|\vec{t}\| = k\|\vec{t}\| + \|\vec{t}\| = \|\vec{v}\| + \|\vec{t}\|$$

(63)

I will redo Exercise 59 in Scratchpad 3

documenting a more natural way to proceed with this "proof" and then comparing it to this solution.

It will be a great way to start Scratchpad 3, actively displaying unrestrained scratch work.